S. S. Montague.

Sam. S. Montague

ON THE

STRENGTH OF MATERIALS.

WORKS BY T. TATE,

OF KNELLER TRAINING COLLEGE, TWICKENHAM,

LATE MATHEMATICAL PROFESSOR AND LECTURER ON CHEMISTY IN THE NATIONAL SOCIETY'S TRAINING COLLEGE, BATTERSEA.

I.

A TREATISE ON THE FIRST PRINCIPLES OF ARITHMETIC, designed for the use of Teachers and Monitors in Elementary Schools. New Edition, with Additions and Improvements. 12mo. Price 1*s*. 6*d*.

II.

ALGEBRA MADE EASY. Chiefly intended for the use of Schools. New Edition. 12mo. Price 2*s*.

III.

EXERCISES ON MECHANICS AND NATURAL PHILOSOPHY; or an easy Introduction to Engineering, for the use of Schools and private Students: containing various Applications of the Principle of Work; the Theory of the Steam-Engine, with simple Machines; Theorems and Problems on Accumulated Work; the Equilibrium of Structure, with the Theory of the Arch; the Pressure and Efflux of Fluids; Calculations on Railway Cuttings, &c. &c. New Edition. 12mo. Price 2*s*.

IV.

THE PRINCIPLES OF GEOMETRY, MENSURATION, TRIGONOMETRY, LAND-SURVEYING, AND LEVELLING; containing familiar Demonstrations and Illustrations of the most important Propositions in Euclid's Elements; Proofs of all the useful Rules and Formulæ in Mensuration and Trigonometry, with their Application to the Solution of Practical Problems in Estimation, Surveying, and Railway Engineering. New Edition, corrected. 12mo. with 317 Woodcuts. Price 3*s*. 6*d*.

V.

THE FIRST THREE BOOKS OF EUCLID'S ELEMENTS OF GEOMETRY, from the Text of Dr. ROBERT SIMSON, together with various useful Theorems and Problems, as Geometrical Exercises on each Book. Second Edition. 12mo. Price 1*s*. 6*d*.

VI.

THE PRINCIPLES OF THE DIFFERENTIAL AND INTEGRAL CALCULUS, simplified and applied to the Solution of various useful Problems in Mathematics and Mechanics. 12mo. Price 4*s*. 6*d*.

VII.

OUTLINES OF EXPERIMENTAL CHEMISTRY; being a familiar Introduction to the Science of Agriculture: designed for the use of Schools and Schoolmasters. Price 1*s*. 6*d*.

VIII.

ON THE STRENGTH OF MATERIALS; containing various Original and Useful Formulæ, specially applied to Tubular Bridges, Wrought Iron and Cast Iron Beams, &c. 8vo.

IX.

THE ELEMENTS OF MECHANISM. 12mo. With Numerous Woodcuts. [*Shortly will be published.*

X.

A TREATISE ON FACTORIAL ANALYSIS AND THE SUMMATION OF SERIES. 8vo. Price 4*s*. (*Published by* G. BELL, 186. *Fleet Street.*)

LONDON: LONGMAN, BROWN, GREEN, AND LONGMANS.

ON THE

STRENGTH OF MATERIALS;

CONTAINING

VARIOUS ORIGINAL AND USEFUL FORMULÆ,

SPECIALLY APPLIED TO

TUBULAR BRIDGES, WROUGHT IRON AND CAST IRON BEAMS, ETC.

BY

THOMAS TATE,

AUTHOR OF

"THE PRINCIPLES OF THE DIFFERENTIAL AND INTEGRAL CALCULUS,

"FACTORIAL ANALYSIS," ETC.

LONDON:

PRINTED FOR

LONGMAN, BROWN, GREEN, AND LONGMANS,

PATERNOSTER-ROW.

1850.

London:
Spottiswoodes and Shaw,
New-street-Square.

INTRODUCTION.

The railway system in this country has recently given rise to a new and important principle of construction,—that of tubular beams; and the facts elicited by the experiments made on this subject have placed in the hands of the mathematician new and valuable data relative to the strength of materials.

Mathematicians had long known, that in order to attain the condition of maximum strength, with a given amount of material, in a beam subjected to transverse strain, the material in the section should be so distributed, that when it is about to yield by the force of compression on the upper side of the beam, it should at the same time be upon the point of yielding to that of extension on the under side. If the resistance of the material to compression were in all cases equal to the resistance which it presents to extension, then it is obvious that the form answering to maximum strength would be that in which the material would be equally accumulated towards the upper and under sides of the beam. But this is not generally the case. At the commencement of the present century, Watt and others, being aware of the fact that cast iron presents a

much greater resistance to compression than it does to extension, constructed cast-iron beams with flanges at the under side in order to compensate for the weakness at that part. Professor Hodgkinson afterwards determined with more precision the relative dimensions of these top and bottom flanges, so as to have them equally strong. On the same principle, hollow beams were found to possess an eligible form of strength. Until very recently, however, the resistance of a combination of wrought-iron plates to transverse strain had not claimed the attention of experimentalists. When Mr. Stephenson first suggested the bold and original idea of a tube, composed of sheet iron, through which railway trains might pass, the practicability of the scheme had to be tested by experiment, and the true form and distribution of the material had to be determined so as to secure a maximum strength with a given section. By a series of inductive experiments, Mr. Fairbairn showed, that such a tube should have the form of a rectangular beam, having the plates at the upper side arranged in the form of cells, as in *Fig.* 10., to counteract the tendency which plates have to crumple when subjected to severe strain, and that the areas of the top and bottom parts should be in the ratio of about 12 to 11. His experiments, as well as those subsequently conducted by Mr. Hodgkinson, also lead to the conclusion, that the plates composing the beam should not be less than about one-half inch in thickness.

In estimating the transverse strength of beams where the material is accumulated in the upper and under

sides, the effect of the rib or plates, as the case may be, connecting these parts may be neglected without incurring any sensible or practical amount of error. On this principle, Mr. Hodgkinson found by experiment, that the strength of cast-iron beams with double flanges varied nearly as the product of the sectional area of the bottom flange multiplied by the depth. Although this formula can scarcely be regarded as a mathematical deduction, yet from the wide range of experiments from which he derived the constant in the formula, it may be taken as sufficiently exact for all beams not widely different in form from those upon which he based his calculations. As applied to tubular beams, this formula is much less empirical. Here the depth of the beam is large as compared with the thickness of the plates or the depth of the bottom part; and this circumstance gives to the formula in this case more of the character and decided impress of a mathematical deduction. (See Art. 59.) Notwithstanding, it must be conceded, that the strict mathematical principle upon which calculations of this kind should be made, is that of the properties of similar beams, as explained in Art. 34. of this work. These properties, in certain restricted cases, were investigated by me in Mr. Fairbairn's work on tubular bridges; but I believe that they are now given, for the first time, in the following Treatise, in their most general and comprehensive form. Various new formulæ are also given throughout the work relative to the strength of different forms of beams, &c.

Isolated, as far as possible, from the influence of

private feelings and sympathies, it is *principles* not *men* which should form the subject of all philosophical discussions. Accordingly, wherever I have presumed to differ from others in opinion, I have done so from a love of truth rather than controversy; and with respect to the three celebrated men associated with the development of one of the boldest and most successful enterprises which has taken place in the history of modern engineering, I have only to state, that I entertain, in common with every one devoted to the study of practical science, a deep sense of the obligations they have conferred on society by their great perseverance and talent.

T. Tate.

Twickenham, July, 1850.

TABLE OF CONTENTS.

	Page
Preliminary Observations and Formulæ	1
Neutral axis of a Beam	4
General Theorem relative to the Neutral Axis	7
Conditions of Rupture	10
General Formulæ of the Moment of Inertia	17
Centres of Compression and Extension	18
To change the Axis of Movements	19
Deflection of Beams	21
GENERAL FORMULÆ RELATIVE TO SIMILAR BEAMS	
Neutral Axis in similar Beams	24
Moment of Inertia of similar Beams	28
Transverse Strength of similar Beams	30
Deflections of similar Beams	37
General Formulæ relative to Beams only in certain respects similar	38
SRENGTH, ETC. OF VARIOUS FORMS OF BEAMS	41
Hollow rectangular Beams	42
Comparison of Strength of solid and hollow Beams	43
Rectangular Cells, maximum Strength	44
To find the Moment of Inertia of Angle-Iron	46
Mr. Fairbairn's Model Tubular Beam	47
Angle-Iron taken into the Calculation	49
Rivet-Holes, do.	51
1. Approximate formula of Strength of Tubular Beams	52
Limits of Error in do.	56
2. Approximate Formulæ	57
Strength, &c. of Beams with Flanges	60
Mr. Hodgkinson's Experiments on cast-iron Beams with double Flanges	65
New Formula for cast-iron Beams	66
Strongest Form of cast-iron Beams	69
STRENGTH, ETC. OF CYLINDRICAL BEAMS	70
Hollow cylindrical Beams, &c.	71
Comparative Strengths of cylindrical and square Beams	73
Comparative Strengths of circular and square Cells	74

Page

Observations relative to the best Form of the Cells in a tubular Beam - - - - - - - - - 75
Why the cellular Structure exhibits such Strength - - - 76
Moment Inertia of a Semicircle, &c. &c. - - - - 77
Strength of a mixed Form of Beam - - - - - 81
STRENGTH, ETC. OF ELLIPTICAL BEAMS, ETC. - - - 82
Hollow elliptical Beams - - - - - - 83
Strength of a mixed Form of Beam - - - - - 85
Moments of Inertia, &c. of triangular and trapezoidal Surfaces - 86
STRENGTH, ETC. OF PARABOLIC BEAMS - - - - 91
Doctrine of similar Beams applied to a few particular Forms - - 94

ON THE

STRENGTH OF MATERIALS.

ON

THE STRENGTH OF MATERIAL.

PRELIMINARY OBSERVATIONS AND FORMULÆ.

1. THE particles of a rigid body are connected together by the force of *cohesion*, and this force must be completely overcome before the body can be broken or ruptured. A force acting upon a rigid body, first tends to change the relative position of its particles, and finally to separate them from each other. The property which bodies possess of taking a new form under the action of a force, and of resuming their original form when the force is withdrawn, is called elasticity. All bodies possess elasticity in a greater or less degree.

2. When the fibres of a beam are pressed together by a force acting longitudinally, the resistance which the material presents is called the resistance of compression. On the contrary, when the fibres of a beam are stretched by a force tending to pull them asunder, the resistance which the material presents is called the force of tension, or the resistance of extension. When the resistance of compression is equal to that of extension, the material in this respect is said to be perfectly elastic, which is nearly the case in bars of wrought iron. But in most kinds of material these forces are different: thus in cast iron the compressive resistance is about $6\frac{1}{2}$ times that of the tensile resistance.

B

+7

3. A beam undergoes transverse strain or rupture when it is broken across, or transversely, as may be the case with joists and beams supporting mason work. Let us suppose that a beam rests upon two supports, as in *fig.* 1. p. 5., and that it has a load placed upon its middle, then this weight causes the beam to bend, and thereby gives rise to a complex action in the fibres of the material: the fibres in the top of the beam become compressed, while those at the bottom part become extended; and there is a certain part in the beam which is neither compressed nor extended; this portion is called the plane of the *neutral axis.* If a beam of timber be cut with a very fine saw about one half through, that is as far as its neutral axis, then it will be found that the strength of the beam is scarcely at all impaired. The position of this axis depends upon the form of the section of the beam, as well as upon the relative resistances of the material to compression and extension. If the transverse section of the beam is rectangular, and the material perfectly elastic, the neutral axis will obviously lie in the central axis of the beam. In general this neutral axis will lie towards the parts where the material presents the greatest resistance, so that the forces on each side of the axis may be duly balanced. Barlow was the first experimentalist, who precisely defined the position of the neutral axis in beams undergoing transverse strain.

Modulus of Elasticity.

4. When the elongation or compression of the fibres of a beam does not exceed a certain limit, they tend to return to the position which they at first had with *a force proportional to the space through which they have been extended or compressed,* as the case may be. But if this elongation or compression be carried beyond a certain point, called *the limit of elasticity,* then the fibres of the beam remain inactively in their new position.

The law of perfect elasticity,—that the amount of extension or compression is proportional to the force,—holds strictly in relation to gases, as expressed in Marriotte's law; but there seems ground for believing that it is only approximately true in solid bodies. However, all our theoretical calculations on the strength of materials are based upon the assumption that the law is applicable to the rigid bodies employed in construction.

The modulus of elasticity is that force E which is necessary to elongate a uniform bar, one square inch section, to double its length (supposing such a thing possible) or to compress it to one-half its length.

Elongation and Compression.

5. Let L be the length of a bar 1 square inch of section, l its elongation or compression with a force of p lbs., then we have from the law of perfect elasticity:

Force to produce an elongation of L in. $=$ E,

$\therefore$ " " " 1 in. $= \frac{E}{L}$,

$\therefore$ " " " l in. $= \frac{El}{L}$;

that is,

$$p = \frac{El}{L} \quad \ldots \ldots \ldots \quad (1.)$$

Now if the bar contains A square inches in the section, then the force P necessary to extend this bar l inches must obviously be A times p:

$$\therefore \quad P = Ap$$

$$= \frac{AEl}{L} \quad \ldots\ldots \quad (2.)$$

or if the value of l be required, we have

$$l = \frac{PL}{AE}. \quad \ldots\ldots\ldots \quad (3.)$$

In these expressions L and l are in the same linear unit.

Example. If, according to Tredgold, the modulus of elasticity of wrought iron be 24900000 lbs., what force will be required to extend a bar 10 ft. long and $\frac{1}{6}$ in. section, through the space of $\frac{1}{2}$ in.?

Here $A = \frac{1}{6}$, $E = 24900000$ lbs., $L = 10 \times 12$, $l = \frac{1}{2}$; hence by eq. (2.),

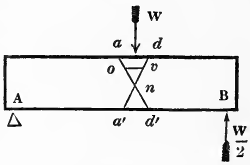
$$P = \frac{\frac{1}{6} \times 24900000 \times \frac{1}{2}}{10 \times 12} = 17300 \text{ lbs. nearly.}$$

THE NEUTRAL AXIS OF A BEAM.

6. From the property of the neutral axis, explained in Art. **3.**, it follows that the sum of the adhesive forces of the fibres resisting rupture on the upper side of this axis must be equal to the sum of the adhesive forces of the fibres on the under side of it. Moreover, as the strain upon the fibres is in proportion to their distances from the plane of the neutral axis, therefore, by Art. **4.**, the resistances of these fibres to extension or compression, as the case may be, will also be in the same proportion.

7. Let AB represent a beam (whose section is made up of rectangles) resting upon the supports A and B, and undergoing transverse strain from the weight W placed upon it. If n be the neutral axis of the section of rupture ad, then the fibres on the upper side nad will undergo compression, while the fibres on the lower side will undergo extension. Let ov be a line of fibres parallel to ad, or, what is the same thing, parallel to the neutral axis of the section; then the resistance of the fibres in ov will be to the resistance of the fibres in ad as no is to na, that is, as their distances from the neutral axis of the section.

Fig. 1.



Let h, h_1 = the respective distances of the neutral axis n from the top and bottom of the beam;

b, b_1 = the respective breadths of the top and bottom parts;

S, S_1 = the compressive and tensile forces respectively exerted by a square inch of the material at the distances h and h_1 from the neutral axis;

s, s_1 = the compressive and tensile forces respectively exerted by a square inch of the material at the distance of unity from the neutral axis;

$x = no$, a variable distance from the neutral axis.

Compressive force per sq. in. at a unit from $n = s$;

∴ " " x units " $= sx$.

Area element of surface at $v = b \Delta x$;

but it has been shown that sx is the resistance of a unit of surface at v,

$\therefore$ Compressive force of this element of surface $= sx \times b \, \Delta x$

$$= sbx \, \Delta x;$$

$\therefore$ Sum of all the compressive forces between ad and ov

$$= sb \sum_x^h x \, \Delta x,$$

$$= sb \int_x^h x \, dx,$$

$$= \frac{sb}{2}(h^2 - x^2).$$

Similarly we have

$$\text{Sum of all the tensile forces} = \frac{s_1 b_1}{2}(h_1^2 - x_1^2).$$

Hence by Art. 6.,

$$\frac{sb}{2}(h^2 - x^2) = \frac{s_1 b_1}{2}(h_1^2 - x_1^2) \quad . \quad . \quad . \quad (4.)$$

Neutral axis of a solid rectangular beam.

8. In eq. (4.), let $x = 0$, $x_1 = 0$, and $b = b_1$; then,

$$sh^2 = s_1 h_1^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.);$$

but, from the principle of elasticity, Art. 4.,

$$s = \frac{S}{h}, \text{ and } s_1 = \frac{S_1}{h_1};$$

therefore, by substitution in eq. (5.),

$$Sh = S_1 h_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.),$$

which expresses the relation of the distances of the neutral axis from the upper and under sides of a rectangular beam.

If the elasticity of the material is perfect, then $s = s_1$; and,

therefore, by eq. (5.), $h = h_1$, that is, in a beam of this kind, the neutral axis lies in the centre of the section.

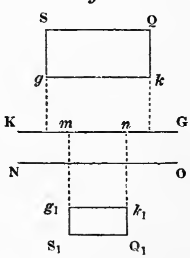
9. *The ratio of the moments of the material undergoing compression and extension about the neutral axis is equal to the ratio of the forces of extension and compression.*

Thus if A be put for the area of the section above the neutral axis, G the distance of its centre of gravity from this axis, and A_1, G_1, the corresponding symbols for the parts below the neutral axis, then

$$\frac{A \cdot G}{A_1 \cdot G_1} = \frac{s_1}{s}.$$

Let $SQkgS_1Q_1k_1g_1$, &c. represent the section of rupture, KG the neutral axis, $Sm = Qn = h$, $gm = kn = e$, $SQ = gk = b$, $S_1m = Q_1n = h_1$, $g_1m = k_1n = e_1$, $S_1Q_1 = g_1k_1 = b_1$; also let $\alpha =$ area Sk, $g =$ the distance of the centre of gravity of Sk from KG, $A =$ the section of the whole material above the neutral axis, G the distance of the centre of gravity of A from the neutral axis, and α_1, g_1, A_1, and G_1 the corresponding dimensions of the parts below the neutral axis: Then putting e for x, eq. (4.) will express the relation from which the position of the neutral axis is determined. By an easy reduction this expression becomes

Fig. 2.



$$s \cdot \frac{h+e}{2} \cdot b\,(h-e) = s_1 \cdot \frac{h_1+e_1}{2} \cdot b_1\,(h_1-e_1);$$

$$\text{but } \frac{h+e}{2} = g,\ b\,(h-e) = \alpha, \text{ and so on};$$

$$\therefore\quad s\alpha g = s_1\alpha_1 g_1.$$

Now, if we regard Sk and S_1k_1, as elementary portions of the section of the beam, the sum of all the elementary moments

$s\alpha g$ will be equal to the sum of all the elementary moments $s_1\alpha_1 g_1$, that is,

$$s\Sigma\alpha g = s_1\Sigma\alpha_1 g_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.),$$

$$\text{but } \Sigma\alpha g = \text{A} \,.\, \text{G, and } \Sigma\alpha_1 g_1 = \text{A}_1 \,.\, \text{G}_1,$$

$$\therefore \quad s \,.\, \text{A} \,.\, \text{G} = s_1 \,.\, \text{A}_1 \,.\, \text{G}_1 \quad . \quad . \quad . \quad . \quad . \quad (8.),$$

$$\therefore \quad \frac{\text{A} \,.\, \text{G}}{\text{A}_1 \,.\, \text{G}_1} = \frac{s_1}{s} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.)$$

10. Substituting $\frac{\text{S}}{h}$ for s, and $\frac{\text{S}_1}{h_1}$ for s_1, eq. (9.) becomes

$$\frac{\text{A} \,.\, \text{G}}{\text{A}_1 \,.\, \text{G}_1} = \frac{h}{h_1} \cdot \frac{\text{S}_1}{\text{S}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.);$$

where h and h_1, in this formula, are the distances of the upper and under edges of the section from the neutral line.

11. Let f and f_1 be put for the respective resistances of the material at the centres of gravity of the top and bottom parts of the section, then

$$f = \text{G} \,.\, s, \text{ and } f_1 = \text{G}_1 \,.\, s_1;$$

therefore, by eq. (8.), we have

$$f\text{A} = f_1\text{A}_1 . \quad . \quad . \quad . \quad . \quad . \quad (11.)$$

12. *When the material in a beam is perfectly elastic the neutral axis passes through the centre of gravity of the section of rupture.*

For in this case $s = s_1$, and then eq. (8.) becomes

$$\text{A} \,.\, \text{G} = \text{A}_1 \,.\, \text{G}_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.);$$

that is, taking the neutral axis as the axis of moments, the moment of the section on the upper side of it is equal to the moment of the section on the under side; therefore, the neutral

axis must pass through the centre of gravity of the whole section.

To find the neutral axis when the elasticity of the material is perfect, that is, when the neutral axis passes through the centre of gravity of the section.

Assume NO (see *fig.* 2. p. 7.) as any convenient axis of moments (as, for example, the centre of the vertical depth of the beam), and let KG be the neutral axis passing through the centre of gravity of the section.

Let K = the area of the whole section,

α, α_1 = the areas of the portions Sk and $S_1 k_1$ respectively,

and g, g_1 = the distances of their respective centres of gravity from NO.

$\bar{x}$ = the distance of NO from KG.

The moment of Sk $= \alpha g$,

and " " " $S_1 k_1 = \alpha_1 g_1$,

$\therefore$ The sum of the moments of the portions above NO $= \Sigma \alpha g$, sum of the moments below NO $= \Sigma \alpha_1 g_1$.

But, because KG passes through the centre of gravity of the section, the moment of the whole section is equal to $K\bar{x}$.

$$\therefore \quad Kx = \Sigma \alpha g - \Sigma \alpha_1 g_1,$$

$$\therefore \quad \bar{x} = \frac{\Sigma \alpha g - \Sigma \alpha_1 g_1}{K}. \quad . \quad . \quad . \quad (13.)$$

13. If $\frac{A}{A_1}$, and $G + G_1$, be the same in two beams, then the neutral axis will have the same position, in the two beams, with respect to the centres of gravity of the upper and under parts of the sections.

For by eq. (12.),

$$\frac{G_1}{G} = \frac{A}{A_1} = \text{a constant},$$

$$\therefore \quad \frac{G + G_1}{G} = \text{a constant};$$

but $G + G_1 = $ a constant.

Therefore G and G_1 are constants, that is, they have the same values for both beams.

14. If $\frac{A}{A_1}$ be a constant ratio in two beams, the ratio of the resistances at the centres of gravity of the top and bottom parts will also be constant.

For by eq. (11.)

$$\frac{f}{f_1} = \frac{A_1}{A},$$

$$\therefore \quad \frac{f'}{f'_1} = \frac{A'_1}{A'},$$

but by assumption $\frac{A_1}{A} = \frac{A'_1}{A'}$,

$$\therefore \quad \frac{f}{f_1} = \frac{f'}{f'_1}.$$

CONDITIONS OF RUPTURE.

15. When rupture is about to take place the beam turns upon the neutral axis *n* of the section of rupture, as a fulcrum (see *fig.* 1. p. 5.); there are, therefore, two forces to be considered, whose moments are in equilibrium with each other, viz., the weight W tending to rupture the beam, and the resistances of

the material on the upper and under sides of the neutral axis in the section of rupture.

16. The weight W (see *fig.* 1.) placed in the middle of the beam, will produce a pressure of $\frac{W}{2}$ upon each of the supports A and B; now, as the beam turns round on n as a fulcrum, this pressure acting at B will have a leverage of half the distance between the supports, or $\frac{l}{2}$; hence the moment of the weight W tending to rupture the beam will be *the pressure acting at* B $\times$ *half the distance between the supports,* or $\frac{W}{2} \times \frac{l}{2} = \frac{Wl}{4}$, which we shall represent by the symbol M. This is called the moment of rupture.

17. It has been explained, Art. **6.**, that the fibres in the section of rupture present different degrees of resistance as they are more or less distant from the neutral axis, and as the beam turns upon this axis the moment of the resistance of any fibre, or its efficacy to prevent rupture, will be its resistance multiplied by its distance from the neutral axis. The sum of all these moments of resistance of the fibres, above as well as below the neutral axis, will be equal to the moment of rupture M.

Taking the figures, notation, &c. of Arts. **7.** and **9.**, we have

Compressive force of an element of surface $= sbx\Delta x$; but this force acts with the leverage x from the neutral axis KG (see *fig.* 2.),

$$\therefore \quad \text{Moment of this element} = sbx^2\Delta x,$$

$\therefore$ Sum of all the moments of the forces of compression between gk and SQ, or moment of the rectangle SQkg

$$= sb\Sigma_e^h x^2 \Delta x,$$

$$= s b \int_e^h x^2 d x,$$

$$= \frac{s b}{3} (h^3 - e^3). \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.)$$

Similarly we have

$$\text{Moment rectangle } S_1 Q_1 k_1 g_1 = \frac{s_1 b_1}{3} (h_1{}^3 - e_1{}^3).$$

Let K_1 = area rectangle $m S Q n$, K_2 = area $m g k n$, $D_1 = S m = Q n$, $D_2 = g m = k n$, k_1 = area $m S_1 Q_1 n$, k_2 = area $m g_1 k_1 n$, $d_1 = S_1 m = Q_1 n$, $d_2 = g_1 m = k_1 n$.

Then the above expressions become

$$\text{Moment rectangle } S Q k g = \frac{s}{3} (K_1 D_1{}^2 - K_2 D_2{}^2). \quad . \quad . \quad (15.)$$

$$\text{Moment rectangle } S_1 Q_1 k_1 g_1 = \frac{s_1}{3} (k_1 d_1{}^2 - k_2 d_2{}^2). \quad . \quad . \quad (16.)$$

Now, if we regard $S k$ and $S_1 k_1$ (*fig.* 2.) as elementary portions of the section, the sum of all the moments on the upper side of the neutral axis added to those on the under side will be equal to the sum of the moments of the resistances to rupture, and therefore equal to the moment of W tending to produce rupture.

∴ Sum of the moments of resistance of the material above

$$K G, \text{ or } m = \frac{s}{3} \Sigma (K_1 D_1{}^2 - K_2 D_2{}^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.)$$

$$= s I,$$

where I is put for $\frac{1}{3} \Sigma (K_1 D_1{}^2 - K_2 D_2{}^2)$, or $\frac{1}{3} \Sigma b (h^3 - e^3)$, which is called THE MOMENT OF INERTIA of the section on the upper side of the neutral axis.

Hence from eqs. (14.) and (15.),

The moment of inertia of the rectangle SQkg about KG as an axis $= \frac{b}{3}(h^3 - e^3)$ or $\frac{1}{3}(\text{K}_1\text{D}^2{}_1 - \text{K}_2\text{D}_2{}^2)$.

If $e = 0$, or Sg becomes Sm, then this expression becomes *the moment of inertia* of the rectangle SQnm about KG as an axis $= \frac{1}{3}bh^3$ or $\frac{1}{3}\text{K}_1\text{D}_1{}^2$.

In like manner, we have

Sum of the moments of resistance of the material below KG, or $m_1 = \frac{s_1}{3}\Sigma\,(k_1 d_1{}^2 - k_2 d_2{}^2)$. . . . . (18.)

$$= s_1 \text{I}_1.$$

But the sum of these moments of resistance is equal to the moment of the force producing rupture,

$$\therefore \text{M} = m + m_1,$$

$$= s\text{I} + s_1\text{I}_1 \quad . \quad . \quad . \quad . \quad . \quad (19.)$$

where $\text{I} = \frac{1}{3}\,\Sigma\,(\text{K}_1\text{D}_1{}^2 - \text{K}_2\text{D}_2{}^2)$, or $\frac{1}{3}\,\Sigma b\,(h^3 - e^3)$ ... (20.)

and $\text{I}_1 = \frac{1}{3}\,\Sigma\,(k_1 d_1{}^2 - k_2 d_2{}^2)$, or $\frac{1}{3}\,\Sigma b_1\,(h_1{}^3 - e_1{}^3)$.... (21.)

These values of I and I_1 depend only upon the form and dimensions of the section of the beam.

Now, when the beam is supported at its extremities and loaded in the middle,

$$\frac{\text{W}l}{4} = \text{M},$$

$$\therefore \text{W} = \frac{4\text{M}}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22.)$$

18. When the elasticity of the material is perfect, or $s_1 = s$,

let $\text{I}_o = \text{I} + \text{I}_1 =$ the moment of inertia of the whole section of the beam, referred to the neutral axis;

$I_x =$ the moment of inertia of the whole section, referred to *any* line NO parallel to the neutral axis, and at the distance $\bar{x}$ from it.

Then eq. (19.) becomes,

$$M = s(I + I_1) = s I_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23.);$$

where I and I_1 are given in eqs. (20.) and (21.).

And from eq. (22.),

$$W = \frac{4M}{l}$$

$$= \frac{4 s I_o}{l},$$

but $S = hs$, where h is put for the distance of the upper edge of the beam from the neutral axis,

$$\therefore W = \frac{4 S I_o}{h l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24.);$$

which is the expression for the breaking weight placed in the middle of the beam, the material being perfectly elastic.

If K_1, D_1, &c., k_1, d_1, &c., in eqs. (20.) and (21.) be taken in reference to the axis NO, then

$$I_x = I + I_1$$

$$= \frac{1}{3}\left\{\Sigma b\,(h^3 - e^3) + \Sigma b_1\,(h_1^3 - e_1^3)\right\} \ldots . (25.)$$

19. *In a solid rectangular beam* $K_2 = 0$, $k_2 = 0$, $I = \frac{1}{3} K_1 D_1^2$, and $I_1 = \frac{1}{3} k_1 d_1^2$,

$$\therefore M = \frac{s}{3} K_1 D_1^2 + \frac{s_1}{3} k_1 d_1^2,$$

but eq. (5.), Art. 8., $s D_1^2 = s_1 d_1^2$,

$$\therefore \text{M} = \frac{s\text{D}_1^2}{3}\left\{\text{K}_1 + k_1\right\}$$

$$= \frac{s\text{D}_1^2}{3} . \text{K}$$

$$= \frac{1}{3} s\text{D}_1 \text{K}, \text{ or } \frac{1}{3} \text{S}_1 d_1 \text{K} \quad . \quad . \quad . \quad (26.);$$

where K is put for the area of the whole section. Hence *the moment of rupture in a rectangular beam varies as the area of the section × the resistance per sq. in. of the material at one edge of the beam × the distance of that edge from the neutral axis.*

When the beam is loaded in the middle, and supported at the extremities, $\text{M} = \frac{\text{W}l}{4}$, therefore eq. (26.) becomes

$$\frac{\text{W}l}{4} = \frac{1}{3} s\text{D}_1 \text{K}, \text{ or } \frac{1}{3} \text{S}_1 \text{D}_1 \text{K};$$

$$\therefore \text{W} = \frac{4 s\text{D}_1 \text{K}}{3l}, \text{ or } \frac{4 \text{S}_1 d_1 \text{K}}{3l} . \quad . \quad . \quad (27.)$$

Now S_1K is the direct tensile strength of the whole section of the beam, therefore the transverse strength of a rectangular beam, loaded in the middle and supported at the extremities, *varies as the direct tensile strength, multiplied by the distance of the neutral axis from the under edge, divided by the distance between the supports.*

20. If the elasticity of the material be perfect, then (Art. **3.**), $\text{D}_1 = \frac{1}{2}\text{D}$, where D is put for the whole depth of the beam; and in this case we have from eq. (26.)

$$\text{M} = \tfrac{1}{6} \text{S} . \text{D} . \text{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28.),$$

and from eq. (27.),

$$\text{W} = \frac{2 \text{S} . \text{D} . \text{K}}{3l} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29.)$$

If the material be incompressible, according to the hypothesis of Galileo and Leibnitz, then the neutral axis will lie at the upper edge of the beam, and therefore $d_1 = \text{D}$, hence eq. (26.) becomes

$$\text{M} = \tfrac{1}{3} \text{S}_1 \,.\, \text{D} \,.\, \text{K.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30.)$$

In like manner, when the material does not admit of extension, we have

$$\text{M} = \tfrac{1}{3} \text{S} \,.\, \text{D} \,.\, \text{K.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31.)$$

The results in eqs. (28.), (30.), and (31.), have been derived on three distinct hypotheses; but in all these expressions it will be observed that *the moment of rupture varies as the area of the section multiplied by the depth.* Mr. Barlow confirmed this law by experiment, and showed that the quantity S, called the modulus of rupture, is a constant for beams of the same material. This constant, or modulus of rupture, has been determined for all the kinds of material employed in construction (see Moseley's Engineering, p. 622.). It has been customary for practical men to assume this law as applicable to beams of all forms in the section. Mr. Hodgkinson showed, by a series of able experiments, that the law was approximately true in reference to cast iron beams with double flanges. Mr. Fairbairn, by an induction of these facts, assumed that the same law would be applicable to the strength of tubular beams; and he accordingly calculated the proportions of the Conway Tube on this assumption (see Mr. Fairbairn's letter to Mr. Stephenson, p. 66. of Mr. F.'s work). Under certain restrictions, the author of this work confirmed Mr. Fairbairn's views by mathematical analysis. A more exact and comprehensive investigation of this subject will be hereafter given in this work.

GENERAL FORMULA OF THE MOMENT OF INERTIA.

21. It has been shown in Art. **17.**, that the moment of inertia of the sum of the rectangular spaces SQkg (see *fig.* 2.) is

$$\text{I} = \tfrac{1}{3}\Sigma\left\{b(h^3 - e^3)\right\}.$$

Now when $e = o$, this expression becomes the moment of inertia of the sum of all the rectangles SQnm; that is,

$$\text{I} = \tfrac{1}{3}\Sigma b h^3. \quad \ldots \ldots \quad (32.)$$

Let RNDB be the section of a beam, undergoing transverse strain; NOB the axis passing through the centre of gravity of the section; SQnm a small rectangular element. Put $y = \text{S}m$, $x = \text{O}m$, and $\Delta x = mn$, then eq. (32.) becomes

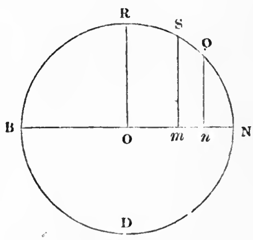


Fig. 3.

$$\text{I} = \tfrac{1}{3}\Sigma y^3 \,\Delta\, x,$$

$$\therefore \quad \text{I} = \tfrac{1}{3}\int y^3 dx \quad \ldots \ldots \quad (33.),$$

where the limits of integration remain to be assigned. Suppose the limits to be taken between $x = 0$, and x, then the moment of inertia of the space ORSm will be

$$\text{I} = \tfrac{1}{3}\int_0^x y^3 dx \quad \ldots \ldots \quad (34.),$$

where y may be determined in terms of x from the equation to the curve RSN.

This expression is equivalent to the following form of double integration,

$$\text{I} = \int_0^x dx \int_0^y y^2 dy,$$

where $dx \int_0^y y^2\,dy$ is the moment of inertia of the element $\text{SQ}nm$, and the whole expression is the sum of the moments of these elements taken between OR and $\text{S}m$, that is, the moment of the space $\text{ORS}m$.

CENTRES OF COMPRESSION AND EXTENSION.

22. DEFINITION. *The centre of compression* is that point in the surface undergoing compression where the whole material may be collected without altering its moment of resistance to rupture. And similarly with respect to the centre of extension.

Let q, q_1 = the distances of the centres of compression and extension respectively from the neutral axis;

m, m_1 = the moments of resistance to rupture of the surfaces of compression and extension;

A, a = the surfaces of compression and extension, and so on to the notation before employed.

By the definition

$$m = \text{SA}q, \text{ and } m_1 = \text{S}_1 a q_1.$$

Hence, in beams of the same material and sectional area, the strength varies as the distances of the centres of compression and extension from the neutral axis.

Now $\text{S} = qs$,

$$\therefore \quad m = \text{SA}q = s\text{A}q^2,$$

$$\text{but } m = s\text{I},$$

$$\therefore \quad s\text{A}q^2 = s\text{I}$$

$$\therefore \quad \text{A}q^2 = \text{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad (35.)$$

and similarly,

$$aq_1^2 = \mathrm{I}_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36.)$$

Hence it appears from the property of the moment of inertia given in works on Mechanics, that I is the moment of inertia of the surface of compression, and that q is the distance of its centre of gyration from the neutral axis; and so on with respect to the centre of extension.

If $s = s_1$, then, putting Q = the distance of gyration of the whole section, K, of the beam, we have

$$\mathrm{M} = \mathrm{SKQ} \quad . \quad . \quad . \quad . \quad . \quad . \quad (37.)$$

$$\text{and } \mathrm{KQ}^2 = \mathrm{I} + \mathrm{I}_1$$

$$= \mathrm{I}_o \text{ or } \mathrm{I}_x \quad . \quad . \quad . \quad . \quad . \quad (38.),$$

as the case may be.

TO CHANGE THE AXIS OF MOMENTS.

23. Let (*fig.* 2.) KG be the neutral axis of the section passing through the centre of gravity, and NO any other axis parallel to KG at the distance $\bar{x}$ from it.

Let $\bar{\mathrm{I}}, \bar{\mathrm{I}}_1$ = the moments of inertia of the sections above and below NO;

$\mathrm{I}_x = \bar{\mathrm{I}} + \bar{\mathrm{I}}_1$, the moment of inertia of the whole section referred to the axis NO.

I_o = the moment of inertia of the whole section referred to KG. The remaining notation being the same as in the preceding investigations.

By eq. (20.) Art. **17.**, we have

$$\mathrm{I} = \Sigma \frac{b}{3} (h^3 - e^3),$$

$$\therefore \quad \overline{\text{I}} = \Sigma\frac{b}{3}\left\{(h+\overline{x})^3-(e+\overline{x})^3\right\};$$

but, by reduction, observing that $a = b\ (h-e)$, and $g=\frac{h+e}{2}$, we have

$$\frac{b}{3}\left\{(h+\overline{x})^3-(e+\overline{x})^3\right\}=\frac{b}{3}(h^3-e^3)+2\overline{x}\,a g+\overline{x}^2 a,$$

$$\therefore \quad \overline{\text{I}} = \Sigma\frac{b}{3}(h^3-e^3)+2\overline{x}\Sigma a g+\overline{x}^2\Sigma a.$$

$$=\text{I}+2\overline{x}\text{A}\,.\,\text{G}+\overline{x}^2\text{A},$$

By substituting $-\overline{x}$, for $\overline{x}$ &c., we have

$$\overline{\text{I}}_1=\text{I}_1-2\,\overline{x}\,\text{A}_1\,\text{G}_1+\overline{x}^2\,\text{A}_1$$

$$\therefore \quad \text{I}_x=\overline{\text{I}}+\overline{\text{I}}_1=\text{I}+\text{I}_1-2\,\overline{x}\,(\text{AG}-\text{A}_1\,\text{G}_1)+\overline{x}^2\,(\text{A}+\text{A}_1),$$

but $\text{I}+\text{I}_1=\text{I}_0$, $\text{A}+\text{A}_1=\text{K}$, and by Art. **12.**, $\text{AG}-\text{A}_1\text{G}_1=0$,

$$\therefore \quad \text{I}_x=\text{I}_0+\overline{x}^2\,\text{K} \quad . \; . \; . \; . \quad (39.)$$

$$\therefore \quad \text{I}_0=\text{I}_x-\overline{x}^2\,\text{K} \quad . \; . \; . \; . \quad (40.);$$

that is, *the moment of inertia of the section taken with respect to the axis passing through the centre of gravity is equal to moment of inertia taken with respect to any other parallel axis, minus the square of the distance between these axes multiplied by the area of the whole section.*

24. Now if I_x be determined with respect to any convenient axis NO (see *fig.* 2.) at the distance $\overline{x}$ from the neutral axis KG, then the moment of inertia I_0 about this neutral axis is given in eq. (40.). Therefore in eq. (24.) the value I_0 is given in eq.

(40.); and, moreover, if d be put for the distance of the axis NO from the upper edge of the beam, we have $h = d \pm \bar{x}$. It is further worthy of observation that the value of $\bar{x}$ is determined in eq. (13.), Art. **12.** Hence,

$$\text{W} = \frac{4\,\text{S}\,\text{I}_0}{h\,l}$$

$$= \frac{4\,\text{S}\,(\text{I}_x - \bar{x}^2\,\text{K})}{l\,(d \pm \bar{x})}. \quad . \quad . \quad . \quad . \quad (41.)$$

DEFLECTION OF BEAMS.

25. Let the annexed figure represent a beam fixed at one extremity, and bent by a weight W suspended from the extremity L; NK, a small portion of the neutral axis, and O its centre of curvature.

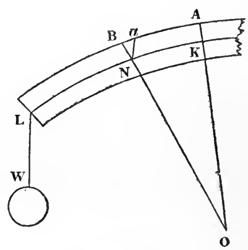


Fig. 4.

Let $c = \text{NK} = \text{A}a$, the length of a small portion of the neutral axis LNK,

$c_1 = \text{B}a$, the extension of the fibre Aa.

$\rho = \text{OK} = \text{ON}$, the radius of curvature of the neutral line NK,

$a = \text{KA} = \text{N}a$, the distance of the upper side of the beam from the neutral axis,

$\text{E} =$ the modulus of elasticity. And so on to other notation as employed in the preceding investigations.

By eq. (1.) Art. 4.,

Force to produce an elongation c_1, or $p = \frac{\mathrm{E} c_1}{c}$;

But the force elongating the fibres at Aa, or $p = sa$;

$$\therefore \quad sa = \frac{\mathrm{E} c_1}{c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (42.)$$

By the similar triangles ONK and NBa,

$$\mathrm{B}a : \mathrm{N}a :: \mathrm{NK} : \mathrm{OK},$$

$$\text{that is, } c_1 : a :: c : \rho,$$

$$\therefore \quad \frac{c_1}{c} = \frac{a}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (43.),$$

substituting in eq. (42.)

$$sa = \frac{\mathrm{E}a}{\rho}$$

$$\therefore \quad s = \frac{\mathrm{E}}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (44.)$$

From eq. (23.), Art. 18.,

$$\mathrm{M} = s\mathrm{I}_0$$

$$= \frac{\mathrm{EI}_0}{\rho},$$

$$\therefore \quad \mathrm{M}\rho = \mathrm{EI}_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (45.)$$

Let HNL represent the curve of deflection, then taking H as the origin of coordinates,

Put $x =$ HV, $y =$ VN, $l =$ HR the length of the beam, and $\delta =$ RL the greatest deflection.

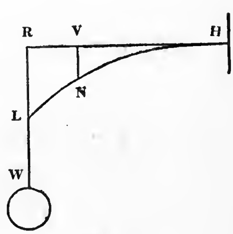


Fig. 5.

The moment of W considered in relation to the point N of the beam is,

$$\text{M} = \text{W} \times \text{RV}$$
$$= \text{W}\,(l - x),$$

therefore by eq. (45.)

$$\text{W}\,(l - x)\,\rho = \text{EI}_0,$$

$$\therefore \quad \frac{1}{\rho} = \frac{\text{W}\,(l - x)}{\text{EI}_0} \quad . \quad . \quad . \quad . \quad . \quad (46.)$$

But $\dfrac{1}{\rho} = \dfrac{\dfrac{d^2y}{dx^2}}{\left(1 + \dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}}$. See the Author's Calculus, p. 215.

Now $\dfrac{dy}{dx}$ is equal to the tangent of the angle which a line touching the curve at N makes with HR; but as the deflection of beams is always small, this angle must also be small, and consequently $\dfrac{dy^2}{dx^2}$ may be neglected.

$$\therefore \quad \frac{1}{\rho} = \frac{d^2y}{dx^2}.$$

Comparing this with eq. (46.), we have

$$\frac{d^2y}{dx^2} = \frac{\text{W}\,(l - x)}{\text{EI}_0}.$$

Successively integrating between the limits x and 0,

$$\frac{dy}{dx} = \frac{\text{W}}{\text{EI}_0}\left(lx - \frac{x^2}{2}\right);$$

$$\text{and } y = \frac{\text{W}}{\text{EI}_0}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) \quad . \quad . \quad . \quad . \quad . \quad (47.),$$

which is the equation of the curve of deflection.*

* For a comprehensive view of the subject of Deflections, see Moseley's "Mechanics of Engineering."

Now when $y = \text{RL} = \delta$, $x = l$,

$$\therefore \quad \delta = \frac{\text{W}}{\text{EI}_0}\left(\frac{l^3}{2} - \frac{l^3}{6}\right)$$

$$= \frac{\text{W}\,l^3}{3\,\text{EI}_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48.);$$

which is the greatest deflection of a beam loaded at one extremity and fixed at the other.

When a beam l between the supports is loaded in the middle (see *fig.* 2.) with a weight W, the deflection takes place at this point as if a force $\frac{\text{W}}{2}$ were applied at the distance $\frac{l}{2}$ from it: hence the greatest deflection in this case will be found by substituting $\frac{\text{W}}{2}$ for W, and $\frac{l}{2}$ for in eq. (48), that is

$$\delta = \frac{\text{W}\,l^3}{48\,\text{EI}_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (49.)$$

GENERAL FORMULÆ RELATIVE TO SIMILAR BEAMS.

NEUTRAL AXIS IN SIMILAR BEAMS.

26. *The neutral axis in similar beams divides the vertical axis proportionally.*

First, suppose the elasticity of the material to be perfect. Assume NO (*fig.* 2. p. 7.) to pass through the centre of the vertical axis of the beam, and KG the neutral axis passing through the centre of gravity of the section.

Let $r =$ the linear ratio of the parts of two beams in all respects similar in section,

$\bar{x}$, a, &c. $=$ the elements of the section of one beam as in Art. **12.**,

x', a', &c. $=$ the corresponding elements in the other beam.

By eq. (13.), Art. **12**,

$$x = \frac{\Sigma a g - \Sigma a_1 g_1}{\mathrm{K}},$$

$$\therefore \quad \bar{x}' = \frac{\Sigma a' g' - \Sigma a'_1 g'_1}{\mathrm{K}'};$$

but $\mathrm{K}' = r^2 \mathrm{K}$, $a' = r^2 a$, $g' = r g$, and so on,

$$\therefore \quad \bar{x}' = \frac{\Sigma r^3 a g - \Sigma r^3 a_1 g_1}{r^2 \mathrm{K}}$$

$$= r . \frac{\Sigma a g - \Sigma a_1 g_1}{\mathrm{K}}$$

$$= r \bar{x}$$

27. Second, suppose the resistance of the material to compression and extension to be unequal. In this case let KG (see *fig.* 2.) be the neutral axis, then adopting the notation of Art. **9.**, we have the following relations:—

$$\frac{\mathrm{A} \cdot \mathrm{G}}{\mathrm{A}_1 \cdot \mathrm{G}_1} = \frac{s_1}{s},$$

$$\text{and} \quad \frac{\mathrm{A}' \cdot \mathrm{G}'}{\mathrm{A}'_1 \cdot \mathrm{G}'_1} = \frac{s'_1}{s'}.$$

Supposing the ratios $\frac{\mathrm{A}_1}{\mathrm{A}}$ and $\frac{s_1}{s}$ to be the same for the two beams, then let us inquire what are the geometrical relations requisite for producing these conditions.

By the assumption $\frac{s_1}{s}=\frac{s_1'}{s'}$, and $\frac{\text{A}}{\text{A}_1}=\frac{\text{A}'}{\text{A}'_1}$, therefore by the two preceding equalities,

$$\frac{\text{G}}{\text{G}_1}=\frac{\text{G}'}{\text{G}'_1}$$

$$\therefore \quad \frac{\overline{\text{G}}}{\overline{\text{G}}'}=\frac{\text{G}}{\text{G}'} \text{ or } \frac{\text{G}_1}{\text{G}'_1} \quad \ldots \ldots \quad (50.),$$

where $\overline{\text{G}}$ is put for $\text{G}+\text{G}_1$ the distance between the centres of gravity of the surfaces of compression and extension, and similarly with respect to $\overline{\text{G}}'$.

Let h, h_1 = the distances of the upper and under edges of the beam from the neutral axis;

$d=h+h_1$ the whole depth of the beam.

$$\frac{s_1}{s}=\frac{\frac{\text{S}_1}{h_1}}{\frac{\text{S}}{h}}=\frac{\text{S}_1}{\text{S}}\cdot\frac{h}{h_1}.$$

Similarly:

$$\frac{s'_1}{s'}=\frac{\text{S}_1}{\text{S}}\cdot\frac{h'}{h'_1},$$

$$\therefore \quad \frac{h}{h_1}=\frac{h'}{h'_1},$$

$$\therefore \quad \frac{d}{d'}=\frac{h}{h'} \text{ or } \frac{h_1}{h'_1}. \quad \ldots \ldots \quad (51.)$$

The formulæ (50.) and (51.) express the geometrical relations involved in the assumptions. But besides these two independent relations we have also $\frac{\text{A}}{\text{A}_1}=\frac{\text{A}'}{\text{A}'}$. Now it is obvious that these three geometrical relations will be fulfilled when the

sections of the beams are similar; and hence eq. (51.) shows that the depths of the beams are divided proportionally by the neutral axis. At the same time it is interesting to observe, that the conditions may not be restricted to exact similarity of form in the sections of the beams.

28. By a similar process of reasoning the first case may be established; for we have by eq. (12.),

$$\frac{\mathrm{A\,G}}{\mathrm{A_1\,G}} = 1, \text{ and } \frac{\mathrm{A'\,G'}}{\mathrm{A'_1G'_1}} = 1,$$

$$\therefore \quad \frac{\mathrm{A\,G}}{\mathrm{A_1\,G_1}} = \frac{\mathrm{A'\,G'}}{\mathrm{A'_1G'_1}}$$

but

$$\frac{\mathrm{A}}{\mathrm{A_1}} = \frac{\mathrm{A'}}{\mathrm{A'_1}},$$

$$\therefore \quad \frac{\mathrm{G}}{\mathrm{G_1}} = \frac{\mathrm{G'}}{\mathrm{G'_1}}$$

$$\therefore \quad \frac{\overline{\mathrm{G}}}{\overline{\mathrm{G}}'} = \frac{\mathrm{G}}{\mathrm{G'}} \text{ or } \frac{\mathrm{G_1}}{\mathrm{G'_1}},$$

which is the same relation as that given in eq. (50.)

Since

$$s \text{ or } s_1 = \frac{\mathrm{S}}{h} = \frac{\mathrm{S_1}}{h_1},$$

$$\therefore \quad \frac{\mathrm{S}}{h} = \frac{\mathrm{S_1}}{h_1},$$

similarly

$$\frac{\mathrm{S}}{h'} = \frac{\mathrm{S_1}}{h'},$$

$$\therefore \quad \frac{h}{h_1} = \frac{h'}{h'_1}$$

$$\therefore \quad \frac{d}{d'} = \frac{h}{h'} \text{ or } \frac{h_1}{h'_1},$$

which is the same relation as that given in eq. (51.).

MOMENTS OF INERTIA OF SIMILAR SURFACES.

29. *The moments of inertia of two similar surfaces, about any axis which divides their vertical depth proportionally, are to each other as the fourth powers of their linear dimensions.*

Let NO (see *fig.* 2. p. 7.) be assumed as any convenient axis of moments, and let KG be the neutral axis of the section passing through the centre of gravity. Suppose NO to divide the vertical depth of the two beams proportionally; then adopting the notation of the preceding formulæ, we have by eq. (20.), Art. **17.**,

$$\mathrm{I}=\frac{1}{3}\Sigma(\mathrm{K}_1\mathrm{D}_1^2-\mathrm{K}_2\mathrm{D}_2^2),$$

where K_1, D_1 &c., are taken in reference to the axis NO.

$$\therefore \quad \mathrm{I}'=\frac{1}{3}\Sigma(\mathrm{K}'_1\mathrm{D}'^2_1-\mathrm{K}'_2\mathrm{D}'^2_2),$$

but $\mathrm{K}'_1=r^2\mathrm{K}_1$, &c., $\mathrm{D}'_1=r\mathrm{D}_1$, &c.; $\therefore$ $\mathrm{K}'_1\mathrm{D}'^2_1=r^4\mathrm{K}_1\mathrm{D}_1^2$, &c.;

$$\therefore \quad \mathrm{I}'=\frac{1}{3}\Sigma(r^4\mathrm{K}_1\mathrm{D}_1^2-r^4\mathrm{K}_2\mathrm{D}_2^2)$$

$$=\frac{r^4}{3}\Sigma(\mathrm{K}_1\mathrm{D}_1^2-\mathrm{K}_2\mathrm{D}_2^2)$$

$$=r^4\mathrm{I}. \qquad (52.)$$

In like manner,

$$\mathrm{I}'_1=r^4\mathrm{I}_1. \qquad (53.)$$

Adding equations (52.) and (53.),

$$\mathrm{I}'+\mathrm{I}'_1=r^4(\mathrm{I}+\mathrm{I}_1),$$

but $I_x = I + I_1$, and $I'_x = I' + I'_1$,

$$\therefore \quad I'_x = r^4 I_x \quad . \quad . \quad . \quad . \quad . \quad . \quad (54.)$$

which is the analytical expression of the Theorem.

Transferring the moments to the axis passing through the centres of gravity, we have by eq. (40.), Art. **23.**

$$I_0 = I_x - \bar{x}^2 K,$$

$$\text{and } I'_0 = I'_x - \bar{x}'^2 K',$$

$$\text{but } I'_x = r^4 I_x, \ \bar{x}' = r\bar{x}, \text{ and } K' = r^2 K,$$

$$\therefore \quad I'_0 = r^4 I_x - r^4 \bar{x}^2 K$$

$$= r^4 (I_x - \bar{x}^2 K)$$

$$= r^4 I_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56.);$$

that is, *the moments of inertia of two similar surfaces, about their centres of gravity as axes, are to each other as the fourth powers of their linear dimensions.*

30. *The distances of the centres of gyration of similar surfaces from their axes of rotation, similarly situated, are proportional to the linear dimensions of the surfaces.*

By eq. (38.), Art. **22.**,

$$K Q^2 = I_x,$$

$$\therefore \quad K' Q'^2 = I'_x,$$

$$\therefore \quad \frac{K' Q'^2}{K Q^2} = \frac{I'_x}{I_x},$$

$$\therefore \quad \frac{r^2 K Q'^2}{K Q^2} = \frac{r^4 I_x}{I_x},$$

$$\therefore \quad \frac{Q'^2}{Q^2} = r^2,$$

$$\therefore \quad \frac{Q'}{Q} = r.$$

which is the analytical expression of the Theorem.

TRANSVERSE STRENGTH OF SIMILAR BEAMS.

31. *The moments of rupture of similar surfaces are to each other as the cubes of their linear dimensions.*

By eq. (17.) Art. **17.**

$$m = s\text{I} = \frac{S\text{I}}{h},$$

$$\therefore \quad m' = \frac{S\text{I}'}{h'},$$

but, Art. **26.**, since the neutral axes divide the surfaces proportionally,

$$\text{I}' = r^4\text{I}, \text{ and } h' = rh,$$

$$\therefore \quad m' = r^3 . \frac{S\text{I}}{h},$$

$$= r^3 m. \quad \ldots \ldots \quad (57.)$$

In like manner,

$$m'_1 = r^3 m_1. \quad \ldots \ldots \quad (58.)$$

Adding equations (57.) and (58.),

$$m' + m'_1 = r^3(m + m_1),$$

but $M' = m' + m'_1$, and $M = m + m_1$,

$$\therefore \quad M' = r^3 M \quad . \quad . \quad . \quad . \quad . \quad . \quad (59.),$$

which is the analytical expression of the Theorem.

Or thus. By eq. (19.) Art. **17.**,

$$M = s I + s_1 I_1 = \frac{S I}{h} + \frac{S_1 I_1}{h_1},$$

$$\therefore \quad M' = \frac{S I'}{h'} + \frac{S_1 I'_1}{h'_1}$$

$$= \frac{S r^4 I}{r h} + \frac{S_1 r^4 I_1}{r h_1}$$

$$= r^3 \left\{ \frac{S I}{h} + \frac{S_1 I_1}{h_1} \right\}$$

$$= r^3 M.$$

32. *The breaking weights of similar beams are to each other as the squares of their linear dimensions.*

By eq. (22.) Art. **17.**,

$$W = \frac{n M}{l},$$

$$\therefore \quad W' = \frac{n M'}{l'},$$

but by eq. (59.) Art. **31.**, $M' = r^3 M$, and $l' = r l$,

$$\therefore \quad W' = \frac{r^3 n M}{r l}$$

$$= r^2 W. \quad . \quad . \quad . \quad . \quad . \quad . \quad (60.)$$

Or thus. Supposing the elasticity of the material to be perfect, we have by eq. (41.) Art. **24.**,

$$\text{W} = \frac{4\,\text{S}(\text{I}_x - \bar{x}^2\text{K})}{l(d \pm \bar{x})}$$

$$\therefore \quad \text{W}' = \frac{4\,\text{S}(\text{I}'_x - \bar{x}'^2\text{K}')}{l'(d' \pm \bar{x}')}$$

But $d' = rd$; by eq. (54.) $\text{I}'_x = r^4\text{I}_x$; by Art. **26.**, $\bar{x}' = r\bar{x}$; $l' = rl$; and $\text{K}' = r^2\text{K}$;

$$\therefore \quad \text{W}' = \frac{4\,\text{S}(r^4\text{I}_x - r^2\bar{x}^2r^2\text{K})}{rl(rd \pm r\bar{x})}$$

$$= \frac{r^2 4\,\text{S}(\text{I}_x - \bar{x}^2\text{K})}{l(d \pm \bar{x})}$$

$$= r^2\text{W}.$$

33. *Strength of the Conway Tube calculated by this formula.*

In the model Conway Tube $\text{W} = 89{\cdot}15$ tons, $l = 900$, and in the Conway tube itself $l = 4800$;

$$\therefore \quad r = \frac{4800}{900} = \frac{16}{3},$$

$$\therefore \quad \text{W}' = r^2\text{W}$$

$$= \left(\frac{16}{3}\right)^2 \times 89{\cdot}15 = 2535 \text{ tons},$$

which is the breaking weight of the Conway Tube, supposing it in all respects similar to the model tube.

In the same manner the strength of any beam (whatever may be the form of its section or the nature of its material) may be calculated from the breaking weight of an experimental beam whose linear dimensions are similar.

34. *The breaking weights of similar beams are to each other as their sectional areas.*

By eq. (60.), Art. **32.**,

$$W' = r^2 W$$

$$\therefore \quad \frac{W'}{W} = r^2,$$

but $A' = r^2 A$, or $\frac{A'}{A} = r^2$,

$$\therefore \quad \frac{W'}{W} = \frac{A'}{A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61.)$$

35. *Strength of the Conway Tube calculated by this formula.*

In the model tube $W = 89{\cdot}15$ tons, A, or area whole section of the material $= 54$, A′, or area section of the Conway tube $= 1530$,

$$\therefore \quad \frac{W'}{89{\cdot}15} = \frac{1530}{54},$$

$$\therefore \quad W' = 2526 \text{ tons.}$$

36. *Strength of Mr. Cubitt's cast-iron beams with double flanges.*

In his best model beam, see *fig.* 11, $CD = 5{\cdot}9$, $CE = {\cdot}84$, $AB = 2{\cdot}72$, $AR = {\cdot}83$, depth $AC = 17\frac{1}{4}$, thickness rib $nv = {\cdot}68$, area section $= 18$, length $= 180$, and $W = 16$ tons. Let it be required to find the breaking weight of a similar beam whose section is 6 sq. in.

Here $W = 16$ tons, $A = 18$, $A' = 6$,

$$\therefore \quad \frac{W'}{16} = \frac{6}{18}, \qquad \therefore \quad W' = 5\tfrac{1}{3} \text{ tons.}$$

37. *The breaking weights of similar beams are to each other as the areas of their bottom sections.*

Let a and a' be put for the areas of the bottom sections of two similar beams, then

$$\frac{A'}{A} = \frac{a'}{a},$$

therefore by eq. (61.)

$$\frac{W'}{W} = \frac{a'}{a}. \quad \ldots \ldots \ldots \quad (62.)$$

38. *In similar beams the cubes of the breaking weights are to each other as the squares of the weights of the beams.*

Let w and w' be put for the weights of the beams respectively, then

$$w' = r^3 w, \qquad \therefore \quad \frac{w'^2}{w^2} = r^6,$$

but by eq. (60.) Art. 32.

$$W' = r^2 W, \qquad \therefore \quad \frac{W'^3}{W^3} = r^6,$$

hence we have by equality,

$$\frac{W'^3}{W^3} = \frac{w'^2}{w^2}. \quad \ldots \ldots \ldots \quad (63.)$$

Let L and L′ be put for the weights of the breaking loads of the two similar beams, then

$$W = L + \frac{w}{2}, \text{ and } W' = L' + \frac{w'}{2},$$

but $W' = r^2 W$, therefore by substitution

$$L' + \frac{w'}{2} = r^2 (L + \frac{w}{2}),$$

substituting $r^3 w$ for w', and reducing,

$$\text{L}' = r^2 \left\{ \text{L} - \frac{w}{2}(r-1) \right\} \quad . \quad . \quad . \quad . \quad (64.),$$

which expresses the breaking load, in terms of the breaking load and weight of the experiment beam.

39. *The breaking weight of beams is equal to the continued product of the sectional area, the depth, and a constant determined by experiment for the particular* FORM *of beam, divided by the distance between the supports.*

Suppose the breaking weight W to be determined by experiment for a beam whose length is l, depth d, and sectional area K. Assume

$$\text{W} = \frac{\text{K}\,d\,\text{C}}{l},$$

where C is a constant, which may be determined from this equation

By eq. (60.) Art. **32.**

$$\begin{aligned} \text{W}' &= r^2 \text{W} \\ &= r^2 \, . \, \frac{\text{K}\,d\,\text{C}}{l} \\ &= \frac{r^2 \text{K} \, . \, rd \, . \, \text{C}}{rl} \\ &= \frac{\text{K}'\,d'\,\text{C}}{l'} \end{aligned}$$

That is the formula,

$$\text{W} = \frac{\text{K}\,d\,\text{C}}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65.),$$

holds true for all similar tubes.

In like manner, if a be put for the sectional area of the bottom portion of the beam, and c a new constant, we have

$$\text{W} = \frac{a\,d\,c}{l}. \quad \ldots \ldots \quad (66.)$$

For $\frac{\text{K}}{a} = r_1$, a constant, $\therefore$ $\text{K} = a\,r_1$, by substituting in eq. (65.) we have

$$\text{W} = \frac{a\,d}{l} \cdot r_1 \text{C}$$

$$= \frac{a\,d\,c}{l},$$

where the new constant c is determined from C by the equality

$$c = r_1 \text{C or } \frac{\text{K C}}{a}. \quad \ldots \ldots \quad (67.)$$

40. Given the areas of the bottom portions of two similar beams, to find the linear ratio, r, of the other parts, &c.

Here $a' = r^2 a$, $\therefore$ $r = \sqrt{\frac{a'}{a}}$;

but $d' = r\,d$,

$$\therefore \quad d' = d\sqrt{\frac{a'}{a}},$$

which is the expression for the depth, and so on to other parts.

Conversely if the depths be given, we have

$$a' = r^2 a, \text{ and } r = \frac{d'}{d},$$

$$\therefore \quad a' = \frac{d'^2}{d^2} \cdot a.$$

41. *The breaking weights of beams of the same length and similar in their sections, are as the cubes of the linear ratio of the sections.*

By eq. (59.), $\text{M}' = r^3 \text{M}$.

Now when the beams have the same length,

$$\text{M}' = \frac{\text{W}' l}{4}, \text{ and } \text{M} = \frac{\text{W} l}{4},$$

$$\therefore \quad \text{W}' = r^3 \text{W}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68.)$$

42. *In beams having similar sections, but different lengths l and l_1 corresponding to the breaking weights* W *and* W_1, *the relation is expressed by the equality*

$$\text{W}_1 = \frac{r^3 \text{W} l}{l_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69.)$$

This result at once follows from eq. (68.), observing that the breaking weights are inversely as the lengths.

Substituting $\sqrt{\frac{a_1}{a}}$ or $\sqrt{\frac{\text{K}_1}{\text{K}}}$ for r, we have

$$\text{W}_1 = \left(\frac{\text{K}_1}{\text{K}}\right)^{\frac{3}{2}} \times \frac{\text{W} l}{l_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70.)$$

These two theorems are useful in comparing the strengths of beams having different lengths and sections. See Art. **78.**

DEFLECTIONS OF SIMILAR BEAMS.

43. *The ultimate deflections of similar beams are to each other as their linear dimensions.*

By eq. (49.), Art. **25.**,

$$\delta = \frac{l^3 \text{W}}{48 \text{EI}}$$

$$\therefore \quad \delta' = \frac{l'^3 \text{W}'}{48 \text{EI}'}.$$

By eq. (56.) $I' = r^4 I$, and by eq. (60.), $W' = r^2 W$, and moreover $l' = r l$,

$$\delta' = \frac{r^3 l^3 \cdot r^2 W}{48 E \cdot r^4 I}$$

$$= r \cdot \frac{l^3 W}{48 E I}$$

$$= r \delta. \qquad (71.)$$

GENERAL FORMULÆ RELATIVE TO BEAMS ONLY IN CERTAIN RESPECTS SIMILAR.

STRENGTH OF BEAMS.

44. Suppose the top and bottom parts of a beam to be composed of rectangular plates, and that the depth of those parts, as well as the thickness of the plates and the whole depth of the beam, to be similar to the corresponding parts of the experimental beam from which the constant C, &c., are determined; to find the breaking weight, &c., when the length is any quantity l_2, and the breadth any quantity b_2, or the area of the section A_2, the sides connecting the top and bottom parts being neglected.

First, let the only variation from similarity in the two beams be in the length. By eq. (22.),

$$W = \frac{nM}{l},$$

$$\therefore \quad W_1 = \frac{nM'}{l_1};$$

but because the two sections are similar, therefore by eq. (59.), Art. 31, $M' = r^3 M$,

$$\therefore \quad W_1 = \frac{r^3 n \, . \, M}{l_1},$$

$$\therefore \quad W_1 = \frac{l}{l_1} \, . \, r^3 W. \quad . \quad . \quad . \quad . \quad . \quad (72.)$$

Or, substituting $\frac{K d C}{l}$ for W, this eq. becomes

$$W_1 = \frac{K' d' C}{l_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (73.)$$

Second, let the breadth, as well as the length, of the beams differ from similarity.

Now, the breaking weights of beams of this kind must evidently be in proportion to their breadths; hence, if b' is put for the breadth of the beam in eq. (72.) and b_2 for the breadth of the beam whose breaking weight we shall call W_2, then we have from eq. (72.), observing that l_1 is changed to l_2 for the sake of uniformity,

$$W_2 = \frac{b_2}{b'} \, . \, \frac{l}{l_2} \, . \, r^3 W,$$

but $b' = rb$,

$$\therefore \quad W_2 = \frac{b_2 l}{b \, l_2} \, . \, r^2 W \; . \quad . \quad . \quad . \quad . \quad (74.).$$

From eq. (73.), changing l_1 to l_2 and K' to K_2,

$$W_2 = \frac{K_2 d' C}{l_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (75.),$$

where K_2 is the sectional area, d' the depth, l_2 the length, and C a constant having the same value as in eq. (65.).

Hence the rule contained in Art 39. applies to beams of any length and breadth, provided that the depths and the structure

of the top and bottom parts are similar. This gives a considerable flexibility to the application of the formula.

It may perhaps be said, that the vertical plates in the top and bottom parts of a tubular beam would interfere with the principle assumed in the preceding investigation, viz. that the sectional area is proportional to the breadth. Now, so far as the strength or theoretical investigation of the general theorem is effected, the vertical plates may be distributed in any manner; hence we may conceive them to be arranged so as to allow the principle assumed to hold strictly true. Hence the general theorem, Art. **39.**, admits of the following important modification.

The breaking weight of a beam, composed of rectangular pieces, is equal to the continued product of the sectional area, the depth, and a constant determined from an experimental beam divided by the distance between the supports, provided that the vertical linear dimensions of the beam are similar to those of the model beam.

45. *Deflection of beams similar in section, but varying in length.*

By eq. (48.) Art. **25.**,

$$\delta = \frac{l^3 \text{W}}{48 \text{EI}},$$

$$\therefore \quad \delta_1 = \frac{l_1{}^3 \text{W}_1}{48 \text{EI}'}.$$

By eq. (72.) Art. **44.**,

$$\text{W}_1 = \frac{l}{l_1} \cdot r^3 \text{W},$$

and since the sections of the beams are similar, therefore, by eq. (56.) Art. **29.**, $\text{I}' = r^4 \text{I}$, hence, by substitution, we have,

$$\delta_1 = \frac{l_1{}^3}{48 \text{E} r^4 \text{I}} \cdot \frac{l}{l_1} \cdot r^3 \text{W},$$

$$\therefore \quad \delta_1 = \frac{l_1^2 l \text{W}}{48 \text{E} r \text{I}}.$$

$$\therefore \quad \frac{\delta}{\delta_1} = r \cdot \frac{l^2}{l_1^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76.)$$

STRENGTHS, &c., OF VARIOUS FORMS OF BEAMS.

64. Let ABCD (see *fig.* 6.) be a solid rectangle in which the neutral axis NO passes through the centre of the section; then by Art. **17.**, p. 13., the moment of inertia of the rectangle ABON is $\frac{1}{3}\text{K}_1\text{D}_1{}^2$; but the moment of the solid rectangle ABCD is double this,

$$\therefore \quad \text{I}_0 = \tfrac{2}{3}\text{K}_1\text{D}_1{}^2$$

$$= \tfrac{1}{3}\text{K}\left(\frac{d}{2}\right)^2$$

$$= \tfrac{1}{12}\text{K}d^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (77.),$$

which is the moment of inertia of a solid rectangle about an axis passing through its centre of gravity, and parallel to the upper or under edge of the section.

Or, putting b for the breadth AB of the beam, this expression becomes

$$\text{I}_0 = \tfrac{1}{12} b d^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (78.)$$

$$\text{Also, M} = \frac{\text{S}\,\text{I}_0}{\frac{1}{2}d}$$

$$= \tfrac{1}{6}\text{S}d\text{K}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (79.)$$

It must be observed that all the expressions given throughout

this work for M, or the moment of rupture, is equal to the moment of the breaking weight W (see Art. **16.**).

HOLLOW RECTANGULAR BEAMS.

47. Let ABCD be the transverse section of a hollow rectangular beam of uniform thickness, NO the axis passing through the centre of the section, then $AN = \frac{1}{2} AD$.

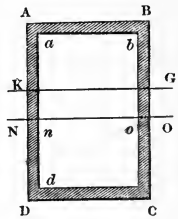


Fig. 6.

Put $d = AD$, the depth of the beam,

$b = AB$, the breadth,

$d_1 = ad$, the interior depth,

$b_1 = ab$, the interior breadth,

$I_0 =$ the moment of inertia of the whole section.

By Art. **17.**, the moment inertia of the rectangle NOBA $= \frac{1}{3} b \left(\frac{d}{2}\right)^3 = \frac{1}{24} b d^3$. Similarly the moment inertia of the interior rectangle $= \frac{1}{24} b_1 d_1^3$.

$\therefore$ Moment inertia section NOAB = moment inertia NOBA − moment inertia $noba = \frac{1}{24}(bd^3 - b_1 d_1^3)$.

But the inertia of the whole is double this expression,

$$\therefore \quad I_0 = \frac{1}{12}(bd^3 - b_1 d_1^3). \quad . \quad . \quad . \quad . \quad . \quad . \quad (80.)$$

When $b = d$, and $b_1 = d_1$, that is, when the section of the beam is square, this equality becomes

$$I_0 = \frac{1}{12}(d^4 - d_1^4). \quad . \quad . \quad . \quad . \quad . \quad (81.)$$

Or, if K be put for the section of the material, we have,

$$\text{I}_0 = \tfrac{1}{12}(d^2 - d_1^2)(d^2 + d_1^2)$$

$$= \frac{\text{K}}{12}(d^2 + d_1^2). \quad . \quad . \quad . \quad . \quad (82.)$$

To find the moment of rupture M of the hollow beam, Art. **47.**, we have,

$$\text{M} = s\,\text{I}_0 = \frac{\text{S}}{\frac{d}{2}} \cdot \text{I}_0,$$

Substituting the value of I_0 given in eq. (80.)

$$\text{M} = \frac{\text{S}}{6\,d}(b\,d^3 - b_1 d_1^3). \quad . \quad . \quad . \quad (83.)$$

And when the beam is square, we have from eq. (82.)

$$\text{M} = \frac{\text{S}\,\text{K}}{6\,d}(d^2 + d_1^2). \quad . \quad . \quad . \quad . \quad (84.)$$

If the thickness of the plates be comparatively small, then $\frac{d_1}{d} = 1$ nearly; hence, in this case,

$$\text{M} = \tfrac{1}{6}\,\text{S}\,\text{K}\,d\left(1 + \frac{d_1^2}{d^2}\right) = \tfrac{1}{3}\,\text{S}\,\text{K}\,d. \quad . \quad . \quad (85.)$$

where *the strength varies as the area of the section multiplied by the depth.*

Comparison of Strength of solid and hollow Beams.

48. In order to compare the strength of a solid beam with a hollow one of the same depth and section of material, we have, by subtracting eq. (79.) from eq. (84.), $\frac{\text{S}\,\text{K}\,d_1^2}{6\,d}$ for the excess of strength of the latter over the former. If the plates composing the hollow beam be thin, then d_1 is nearly equal to d, and therefore, in this case, the hollow beam will have double the strength of the solid one.

49. *To determine the moment of inertia, &c., of a square rectangular cell* ABCD, *about any axis* NO *parallel to* DC, *or* AB.

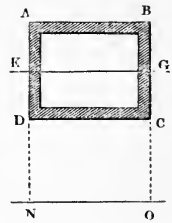


Fig. 7.

By eq. (82.), Art. **47.**, the moment of inertia of this cell about an axis KG passing through its centre of gravity is

$$I_0 = \frac{K}{12}(d^2 + d_1^2).$$

And by eq. (39.), Art. **23.**,

$$I_x = I_0 + \bar{x}^2 K$$

$$= \frac{K}{12}(d^2 + d_1^2 + 12\,\bar{x}^2) \quad . \quad . \quad . \quad (86.),$$

which is the expression required, $\bar{x}$ being the distance between KG and NO, or the distance of the centre of gravity of the cell from the axis NO.

50. *To find the moment of inertia of a rectangular cell* ABCD, *about any axis* NO *parallel to* DC *or* AB (see last figure).

Here, substituting the value of I_0 given in eq. (80.), we have

$$I_x = I_0 + \bar{x}^2 K$$

$$= \tfrac{1}{12}(b\,d^3 - b_1 d_1^3) + \bar{x}^2 K. \quad . \quad . \quad . \quad (87.)$$

This expression will also give moment of inertia of a series of cells ABSR (see *fig.* 10.) about KG; where $b =$ AB the exterior breadth, $d =$ AR the exterior depth, $b_1 =$ the whole breadth of the spaces in the cells, $d_1 =$ the interior depth of the cells, $K =$ the area of the section of the material, and $\bar{x} =$ the distance of the centre of the cells from the axis KG.

51. *To determine the conditions of maximum strength;* b, d, $\bar{x}$, and K being constant.

Here I_x, and, consequently, M_x, will be a maximum when $b_1 d_1^3$ is a minimum, or putting K_1 for the area of the internal spaces,

$$b_1 d_1^3 = K_1 d_1^2 = \text{a minimum},$$

or d_1, a minimum, that is, when the material is accumulated in the two horizontal plates AB and RS. However, in practice, it is necessary that the vertical plates should have a certain thickness to counteract the tendency which the horizontal plates have to *crumple.*

52. *To find the strength, &c., of a hollow beam, when the material is not equally distributed on each side of the neutral axis.*

Let KG (see *fig.* 6.) be drawn bisecting the vertical depth of the beam, and parallel to the neutral axis NO.

Let K_1, K_2 = the areas of the external and internal rectangles above KG; D_1 and D_2 being the respective depths of these rectangles estimated from KG.

k_1, k_2, d_1, d_2 = the corresponding dimensions below KG.

Then we have for the position of the neutral axis (see Art. **12.**)

$$\bar{x} = \frac{K_1 D_1 - K_2 D_2 - (k_1 d_1 - k_2 d_2)}{2K} \quad . \quad . \quad (88.),$$

which expresses the distance of the neutral axis NO from the central line KG.

By Art. **17.**, we have for the moment of inertia I_x about the axis KG,

$$I_x = \frac{1}{3}\left\{ K_1 D_1^2 - K_2 D_2^2 + k_1 d_1^2 - k_2 d_2^2 \right\}. \quad . \quad . \quad (89.)$$

Moreover, for the moment of inertia referred to the neutral axis, we have by eq. (40.), Art. **23.**,

$$\mathrm{I}_0 = \mathrm{I}_x - x^2\,\mathrm{K}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90.)$$

And, finally,

$$\mathrm{M} = s\,\mathrm{I}_0 = \frac{\mathrm{S}\,\mathrm{I}_0}{\mathrm{D}_1 + \bar{x}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91.),$$

where the value of $\bar{x}$ is given in eq. (88.), and I_0 in eqs. (89.) and (90.). In these expressions $\mathrm{D}_1 = d_1 = \frac{d}{2}$, or one-half the whole depth of the beam.

53. *To find the moment of inertia of angle-iron.*

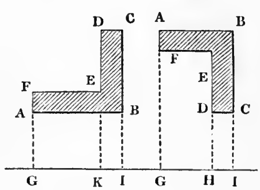
Let ABCD be the section of the angle-iron, and GI the axis of moments.

Let $a = \mathrm{BI} = \mathrm{AG}$, the distance of the edge AB from GI;

$b = \mathrm{AB} = \mathrm{BC}$, the length of the angle-iron;

$t = \mathrm{AF} = \mathrm{DC}$, the thickness of the angle-iron.

Fig. 8. *Fig.* 9.



Then in *fig.* 9., we have, by Art. **17.**,

I = moment inertia rectangle ABIG − moment inertia rectangle FEHG − moment inertia rectangle DCIH.

$$= \frac{1}{3}\left\{ba^3 - (b-t)\,(a-t)^3 - t\,(a-b)^3\right\}. \quad . \quad (92.)$$

Similarly in *fig.* 8., we have

$$\mathrm{I} = \frac{1}{3}\left\{(b-t)\,(a+t)^3 + t\,(a+b)^3 - ba^3\right\}. \quad . \quad (93.)$$

In eq. (92.) if $a = b$, or $\mathrm{CI} = 0$, then

$$\mathrm{I} = \frac{1}{3}\left\{b^4 - (b-t)^4\right\}. \quad . \quad . \quad . \quad . \quad . \quad (94.)$$

In eq. (93.) if $a = 0$, or $\text{AG} = \text{BI} = 0$, then

$$\text{I} = \frac{1}{3}\left\{(b - t)\ t^3 + tb^3\right\}. \quad . \quad . \quad . \quad (95.)$$

TO DETERMINE THE STRENGTH, ETC. OF A TUBULAR BEAM ABDC, COMPOSED OF SQUARE CELLS, AS, IN THE TOP PART, AND OF THICK PLATES, CD, IN THE BOTTOM PART, SIMILAR TO THE MODEL TUBE EXPERIMENTED UPON BY MR. FAIRBAIRN.

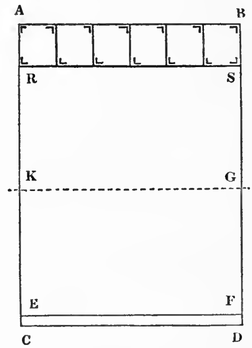


Fig. 10.

54. *To find the neutral axis, the elasticity of the material being perfect.*

Let $a =$ the area of the material in the top, AS;

$\beta =$ the area of the bottom plates.

$\gamma =$ the area of the two side plates, RE and SF;

$a_1 =$ the distance of the centre of gravity of the top cells from the lower edge, CD;

$\beta_1 =$ half the thickness of the plates in CD;

$\gamma_1 =$ the distance of the centre of gravity of the side plates from CD;

$\text{K} =$ area of the whole section ABDC;

$\bar{x} = \text{KC} = \text{GD}$, the distance of the neutral axis from CD.

Then taking CD as the axis of moments, we have, Art. **12.**,

$$\bar{x} = \frac{\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1}{\text{K}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (96.)$$

55. In the model tube, $\alpha = 24$, $\beta = 22$, $\gamma = 7$, $\alpha_1 = 50{\cdot}7$, $\beta_1 = {\cdot}3$, $\gamma_1 = 24$, and $\kappa = 53$,

$$\therefore \quad \bar{x} = \frac{24 \times 50{\cdot}7 + 22 \times {\cdot}3 + 7 \times 24}{53} = 26{\cdot}2.$$

Now, as the whole depth of the beam is 54, we have the distance of the neutral axis from the upper edge, viz. AK $= 54 - 26{\cdot}2 = 27{\cdot}8$. Hence, it appears that the neutral axis passes nearly through the centre of the beam.

56. *To find the moments of rupture, &c.*

Let $h =$ AK $=$ BG, the distance of the upper edge from the neutral axis KG;

$h_1 =$ CK $=$ DG, the distance of the lower edge from KG;

$p =$ the whole breadth of the spaces in the top cells;

$e =$ AR $=$ BS, the depth of the top cells;

$k =$ the thickness of the plates in the top cells;

$b =$ AB $=$ CD, the breadth of the tube;

$k_1 =$ the thickness of the bottom plates;

$k_2 =$ the sum of the thicknesses of the side plates, RE and SF.

$I_0 =$ moment ABGK $-$ moment space KGSR $-$ moment space in the cells AS $+$ moment KGDC $-$ moment space KGFE.

Now moment ABGK $= \frac{1}{3} b h^3$, moment space KGSR $= \frac{1}{3}(b - k_2)(h - e)^3$, moment cells AS $= \frac{1}{3} p \left\{(h-k)^3 - (h - e + k)^3\right\}$,

moment KGDC $=\frac{1}{3}bh_1^3$, moment space KGFE $=\frac{1}{3}(b-k_2)(h_1-k_1)^3$. Hence we have, by substitution and reduction,

$$\mathrm{I}_0=\frac{1}{3}\Big[b(h^3+h_1^3)-(b-k_2)\left\{(h-e)^3+(h_1-k_1)^3\right\}$$

$$-p\left\{(h-k)^3-(h-e+k)^3\right\}\Big] \quad \ldots\ldots \quad (97.),$$

which expresses the moment of inertia of the whole section.

By substituting this value for I_0 in eq. (24.), Art. **18.**, the expression for the breaking weight, &c. is determined.

57. If the angle-iron is to be taken into the calculation, the following method of investigation may be adopted.

First, let us calculate the moment of inertia of the top-cells AS, about an axis, passing through the centre of the cells, parallel to AB or RS.

Let e, e' = the exterior and interior depth of the top cells respectively,

b, b' = the exterior and interior breadth of the same (see observations, Art. **50.**),

a' = the distance of the centre of the top cells from KG,

a = the area of the material in the top cells,

r = the length of the angle iron,

t = the thickness of the angle iron,

n = the total number of angle irons in the top part,

q = the moment of inertia of the top part about its axis,

Q = the moment of the same about KG.

These symbols being accented for the corresponding dimensions below the neutral axis, and the other notation being the same as in the preceding investigations, we have, by eq. (92.) Art. **53.**,

Moment of inertia of the angle iron in the top cells, about a line passing through their centre,

$$=\frac{n}{3}\left\{r\frac{e'^3}{8}-(r-t)\left(\frac{e'}{2}-t\right)^3-t\left(\frac{e'}{2}-r\right)^3\right\}$$

$$=\frac{n}{24}\left\{re'^3-(r-t)\left(e'-2t\right)^3-t\left(e'-2r\right)^3\right\}. \quad (98.)$$

By eq. (80.) Art. **47.**, moment of inertia of the plates in the top cells

$$=\tfrac{1}{12}\{be^3-b'e'^3\}. \quad \ldots\ldots \quad (99.)$$

Adding eq. s. (98.) and (99.),

$$q=\frac{n}{24}\left\{re'^3-(r-t)(e'-2t)^3-t(e'-2r)^3\right\}+\tfrac{1}{12}(be^3-b'e'^3). \quad (100.)$$

In order to refer this moment to the axis KG, we have, by eq. (39.), Art. **23.**,

$$Q=q+\alpha'^2\alpha. \quad \ldots\ldots \quad (101.)$$

Moment inertia side plates $=\frac{1}{3}k_2\,(h-e)^3$

$$\therefore \quad I=Q+\tfrac{1}{3}k_2\,(h-e)^3. \quad \ldots \quad (102.)$$

Supposing the bottom part of the tube to be constructed with cells, as in the Conway and Britannia tubes, we similarly have

$$Q_1=q_1+\alpha_1'^2\alpha_1 \quad \ldots\ldots \quad (103.)$$

$$I_1=Q_1+\tfrac{1}{3}k_2\,(h_1-e_1)^3. \quad \ldots \quad (104.)$$

Hence we have for moment of rupture of the whole section, Art. **18.**,

$$\text{M} = \frac{\text{S}}{h}(\text{I} + \text{I}_1). \quad . \quad . \quad . \quad . \quad (105.)$$

Where, if S be given, M, as well as W, may be determined; or if $\text{M} = \frac{\text{W}\,l}{4}$ be determined by experiment, then S may be found from eq. (105.); and the value of h may be found from eq. (96.), where β must be taken for the area of the material in the bottom cells, and β_1 for the distance of their centre of gravity from KG.

58. It is desirable that an allowance should be made for the rivet holes of the angle iron; for this purpose, let there be two rivet holes in each arm of the angle iron, and

Let c = the diameter of the rivet holes,

v, v' = the distances of the edges of the rivet holes, in the vertical plates, at the greatest distance from the central axis of the top cells;

q' = moment of inertia of all the rivet holes, in the top cells, about the central axis of the cells.

Moment inertia rivet holes in the horizontal plates

$$= \frac{2n}{3}\left\{c \cdot \frac{e^3}{8} - c\left(\frac{e}{2} - k - t\right)^3\right\} = \frac{nc}{12}\left\{e^3 - (e - 2k - 2t)^3\right\}.$$

Moment inertia rivet holes in the vertical plates

$$= \frac{n}{3}\left\{(k+t)\,v^3 - (k+t)\,(v-c)^3 + (k+t)\,v'^3 - (k+t)\,(v'-c)^3\right\}$$

$$= \frac{n\,(k+t)}{3}\left\{v^3 - (v-c)^3 + v'^3 - (v'-c)^3\right\}.$$

$$\therefore \quad q' = \frac{nc}{12}\left\{e^3 - (e - 2k - 2t)^3\right\} + \frac{n\,(k+t)}{3}\left\{v^3 - (v-c)^3 + v'^2 - (v'-c)^3\right\} \quad (106.)$$

If there is only one rivet in each arm of the angle iron, then

$$q^1 = \frac{nc}{24}\left\{e^3 - (e-2k-2t)^3\right\} + \frac{n(k+t)}{3}\left\{v^3-(v-c)^3\right\}.$$

Therefore, in the place of eq. (101.), we must in this case employ the following expression for Q, viz.

$$Q = q - q' + \alpha'^2 \alpha. \quad . \quad . \quad . \quad . \quad . \quad (107.)$$

And so on similarly for the value of Q_1.

1. *Approximate Formula.*

59. *In tubular beams, as they are now usually constructed, the strength nearly varies as the area of the top or bottom part multiplied by the depth divided by the distance between the supports.*

That is, if

a, a_1 = the top and bottom areas respectively,

d = the depth of the beam,

W = the breaking weight,

l = the distance between the supports,

C, C_1 = constants determined from experiment on a tube constructed on the peculiar principle of tubular beams; then

$$W = \frac{adC}{l} \text{ or } \frac{a_1 d C_1}{l}.$$

Let e = AR = BS, the depth of the top cells (see *fig.* 10.);

g = the distance of the centre of the top cells from the neutral axis KG;

b = the breadth of the material in the top cells, supposing it collected in vertical plates;

h = the distance of the upper edge of the beam from the neutral axis; and so on.

Without infringing upon the peculiarity of the structure, we may suppose that the material in the top and bottom cells is collected in vertical plates, then, neglecting the material in the side plates KR and GS, we have, Art. **17.**, eq. (14.),

$$m = \frac{sb}{3}\left\{\left(g+\frac{e}{2}\right)^3 - \left(g-\frac{e}{2}\right)^3\right\}$$

$$= \frac{sb}{3}\left\{3g^2e + \frac{e^3}{4}\right\}$$

$$= sbeg^2\left\{1 + \frac{e^2}{12g^2}\right\},$$

neglecting $\frac{e^2}{12g^2}$ as being very small, and also observing that $be = a$, we have

$$m = sag^2.$$

In like manner,

$$m_1 = s_1 a_1 g_1^2,$$

$$\therefore \quad \mathrm{M} = m + m_1$$

$$= sag^2 + s_1 a_1 g_1^2;$$

but by eq. (8.) Art. **9.**,

$$sag = s_1 a_1 g_1,$$

$$\therefore \quad \mathrm{M} = sag(g+g_1) \text{ or } s_1 a_1 g_1 (g+g_1)$$

$$= sag\mathrm{G} \text{ or } s_1 a_1 g_1 \mathrm{G}$$

$$= \frac{\mathrm{S}}{h} ag\mathrm{G} \text{ or } \frac{\mathrm{S}_1}{h_1} a_1 g_1 \mathrm{G} \quad \ldots\ldots \quad (108.)$$

$$= \mathrm{S}ad \times \frac{g\mathrm{G}}{hd} \text{ or } \mathrm{S}_1 a_1 d \times \frac{g_1\mathrm{G}}{h_1 d},$$

taking $\frac{gG}{hd}$ and $\frac{g_1G}{h_1d}$ as constants, each of them being, in all practical cases, nearly equal to unity, we have

$$\text{M} = \text{S}ad \text{ or } \text{S}_1 a_1 d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109.),$$

where the strength varies as the product of the top or bottom areas multiplied by the depth.

When the beam is supported at each end and loaded in the middle,

$$\frac{\text{W}l}{4} = \text{M} = \text{S}ad \text{ or } \text{S}_1 a_1 d$$

$$\text{W} = \frac{4\text{S}ad}{l} \text{ or } \frac{4\text{S}_1 a_1 d}{l}$$

$$\therefore \quad \text{W} = \frac{ad\text{C}}{l} \text{ or } \frac{a_1 d\text{C}_1}{l} \quad . \quad . \quad . \quad (110.),$$

where the values of C and C_1 are determined as follows:—

60. In the model Conway tube, $l = 900$, $a_1 = 22{\cdot}5$, $d = 54$, and $\text{W} = 89{\cdot}15$ tons, therefore from eq. (110.)

$$\text{C}_1 = \frac{l\text{W}}{a_1 d}$$

$$= \frac{900 \times 89{\cdot}15}{22{\cdot}5 \times 54}$$

$$= 66 \text{ tons.}$$

Or making a deduction of 3·5 inches for the rivet holes in the bottom area,

$$\text{C}_1 = \frac{900 \times 89{\cdot}15}{19 \times 54} = 78 \text{ tons.}$$

Substituting the value of the first constant in eq. (110.), we have

$$\text{W} = \frac{66 a_1 d}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (111.),$$

which expresses the breaking weight, in tons, of any tubular beam loaded in the middle and supported at the extremities, a_1 being the area of the bottom part of the beam in inches, d the whole depth of the beam, and l the distance between the two supports expressed in the same units as d.

Cor. From eq. (110.), we have,

$$4\text{S} = \text{C}, \therefore \text{S} = \tfrac{1}{4}\text{C}.$$

61. *Strength of the Conway Tube calculated by formula* (111.)

Here $a_1 = 517$, $d = 302$, and $l = 4800$,

$$\therefore \quad \text{W} = \frac{66 a_1 d}{l}$$

$$= \frac{66 \times 517 \times 302}{4800} = 2146 \text{ tons.}$$

62. *When the depth and distance between the supports are the same in two beams, the breaking weights are as the areas of the top or bottom parts.*

This property at once follows from eq. (110.)

The truth of this theorem is confirmed by Mr. Fairbairn's experiments on the model tube. From experiments 1, 3, 4, and 10 on the model tubular beam, as given in his work on Tubular Bridges, we have the following table of results: —

No. of Experiments.	Area, bottom.	Breaking Weight.	Proportional Area to Breaking Weight.
1	6·8	79,578	1 : 91
3	12·8	126,128	1 : 98
4	17·8	148,129	1 : 84
10	22·45	192,892	1 : 86

Here the law almost exactly obtains in the two last experiments. In the two first cases the breaking weights are slightly in excess of that which would be indicated by the law; but this might have been anticipated in conformity with the relation $W = \frac{adc}{l}$, since the areas of the tops of these tubes are in excess of the proportion which they should have to the bottom areas. However, every candid person cannot fail of recognising the preceding formula as being a manifest induction from these experiments.

Limits of Error involved in Formula (110.)

63. First with respect to the quantity $\frac{e^2}{12g^2}$; in the model tube $e = 6{\cdot}5$, by Art. **55.**, $AK = 27{\cdot}8$,

$$\therefore \quad g > 27 - 3{\cdot}2 > 23{\cdot}8, \qquad \therefore \quad \frac{e^2}{12g^2} < \frac{6{\cdot}5^2}{12 \times 23{\cdot}8^2} < \frac{1}{160}.$$

Hence this portion of the formula which is rejected is less than $\frac{1}{160}$th part of that which is retained, so that the relation, $M = sagG$, or $s_1 a_1 g_1 G$, is almost mathematically exact.

Second, with respect to the quantity $\frac{g_1 G}{h_1 d}$, in the model tube, $e = 6{\cdot}5$, $e_1 = {\cdot}6$, $d = 54$, $h_1 = 26{\cdot}2$,

$$g_1 = 26{\cdot}2 - \frac{{\cdot}6}{2} = 26, G = g + g_1 = 51, \text{ nearly,}$$

$$\therefore \quad \frac{g_1 G}{h_1 d} = \frac{26}{26{\cdot}2} \cdot \frac{51}{54} = \frac{25}{27} = 1 \text{ nearly,}$$

Hence there is an error of only about $\frac{1}{13}$th occasioned by the assumption of the formula (109.)

64. In the tubular beams which Mr. Fairbairn has recently constructed, the depth of the beams are increased in ratio; hence in such cases the error is less than that determined in the foregoing calculation.

The friends of Mr. Stephenson, actuated by an intemperate zeal for their distinguished patron, have publicly declared that this important formula is empirical. It would certainly better serve their cause if they could sustain their declamation by an appeal to mathematical analysis. The fact is, the formula is as nearly true as any general formula of the kind can profess to be. It will be hereafter shown, that it is a nearer approximation to truth than Mr. Hodgkinson's celebrated formula relative to cast-iron beams.

2. *Approximate Formula.*

65. In beams where the material is properly distributed, we should always have $\frac{a}{a_1} = \frac{s_1}{s}$, or $s\,a = s_1\,a_1$; in this case, therefore, we have, from eq. (8.),

$$g = g_1 \, . \quad . \quad . \quad . \quad . \quad . \quad (112.);$$

that is, *the distances of the centres of gravity of the top and bottom parts of a beam, from the neutral axis, are equal to one another.*

Hence we have, from eq. (108.),

$$\text{M} = \frac{\text{S}}{h} \, . \, a g \text{G}$$

$$= \frac{\text{S}}{\frac{\text{G}}{2} + \frac{e}{2}} \, . \, a \, . \, \tfrac{1}{2}\,\text{G} \, . \, \text{G}$$

$$= \frac{\text{S}\,a\,\text{G}^2}{\text{G} + e} \text{ or } \frac{\text{S}_1\,a_1\,\text{G}^2}{\text{G} + e_1} . \quad . \quad . \quad . \quad . \quad (113.)$$

Therefore in beams loaded in the middle, and supported at the two extremities,

$$\mathrm{W} = \frac{4\mathrm{S}a\mathrm{G}^2}{l(\mathrm{G}+e)} \text{ or } \frac{4\mathrm{S}_1 a_1 \mathrm{G}^2}{l(\mathrm{G}+e_1)} \quad . \quad . \quad . \quad (114.),$$

where the *very small* quantity $\frac{e^2}{12g^2}$ is all that is neglected,

$$\therefore \quad \mathrm{S}_1 = \frac{\mathrm{W}l(\mathrm{G}+e_1)}{4a_1\mathrm{G}^2} \quad . \quad . \quad . \quad . \quad (115.)$$

These formulæ are applicable to cast-iron beams with double flanges, or to any beams of this form, whatever may be the nature of the material, or the relation between s and s_1, provided the beam is properly constructed.

In the Conway model tube, $d = 54$, $e = 6{\cdot}5$, $e_1 = {\cdot}6$, $a_1 = 22{\cdot}5$, $l = 900$, $\mathrm{W} = 89{\cdot}15$ tons; $\therefore \mathrm{G} = 54 - \frac{6{\cdot}5 + {\cdot}6}{2} = 50{\cdot}5$,

$$\therefore \quad \mathrm{S}_1 = \frac{89{\cdot}15 \times 900 \times (50{\cdot}5 + {\cdot}6)}{4 \times 22{\cdot}5 \times 50{\cdot}5^2} = 18 \text{ tons nearly.}$$

And if $3\frac{1}{2}$ inches be deducted from the bottom area on account of the rivet holes, we find

$$\mathrm{S}_1 = 21 \text{ tons.}$$

66. *Strength of the Conway Tube calculated by formula* (114.)

Here $a_1 = 517$, $e = e_1 = 22$, $d = 302$, $l = 4800$; by eq. (115.) $\mathrm{S}_1 = 18$ tons; $\therefore \mathrm{G} = 302 - 22 = 280$,

$$\therefore \quad \mathrm{W} = \frac{4\mathrm{S}_1 a_1 \mathrm{G}^2}{l(\mathrm{G}+e_1)} = \frac{4 \times 18 \times 517 \times 280^2}{4800 \times 302}$$

$$= 2013 \text{ tons,}$$

which very nearly coincides with the result obtained by formula (111.) Art. **61.**

67. In the preceding calculations it has been assumed that a tubular beam follows the same laws of transverse strain as an ordinary solid beam.

The model tubular beam constructed by Mr. Fairbairn gave the true form and elements of stability of the Conway and Menai Bridges; and the result of his experiments upon this beam will doubtless be regarded by a future age, if not by the present, as one of the most important experimental facts, relative to the strength of material, which has hitherto been discovered.

Now in this model beam, the principle of crumpling seems to be eliminated by the thickness given to the plates, by the combination of the cells, and by the *strong* angle irons used in connecting the plates. This is rendered apparent from the fact, that the top area is nearly equal to the bottom one, when the equality of resistance is attained. Hence the model tubular beam may be regarded as a common beam obeying the ordinary laws of compression and extension when subjected to transverse strain. The assumption, therefore, that the Conway Tube will have the same resistance to compression in its top structure as the thin rectangular cells experimented upon by Mr. Hodgkinson, is erroneous in principle; and this is rendered still more apparent from the calculations on the model tube, given in Art. **65.**, where the resistance per sq. in. to compression is found to be about 18 tons in the place of 8 tons, which Mr. Hodgkinson assigns to it. It would further appear, that when a beam of this kind is broken by transverse strain, the material in the top cells undergoes a more complex strain than that exhibited by a simple crushing force, where the material is not allowed to bend under the pressures applied to it. Without incurring a great amount of error, the model tube may be treated on the assumption that the material is perfectly elastic.

STRENGTH, ETC. OF BEAMS WITH DOUBLE FLANGES.

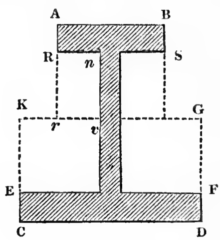


Fig. 11.

68. Let A B D C be the section of the beam, ABSR the top flange, EFDC the bottom one, nv the vertical rib, and KG the neutral axis.

The neutral Axis.

Let $\text{A}, \text{A}_1 =$ area sections above and below KG respectively;

$\text{G}', \text{G}'_1 =$ the distances of the centres of gravity of the top and bottom sections from KG;

$e, e_1 =$ AR and CE respectively;

$b, b_1 =$ AB and CD respectively;

$a, a_1 =$ areas AS and ED respectively;

$h, h_1 =$ the distance of AB and CD respectively from KG;

$k_2 =$ thickness of the vertical rib;

$d =$ the whole depth of the beam; and so on.

By eq. (9.), Art. **9.**,

$$\frac{\text{A}\,\text{G}'}{\text{A}_1\text{G}'_1} = \frac{s_1}{s}. \quad \ldots \ldots \quad (116.)$$

Now $\text{A}\,\text{G}' = a\left(h - \frac{e}{2}\right) + \frac{1}{2}k_2(h-e)^2$,

$$A_1 G'_1 = a_1\left(d - h - \frac{e_1}{2}\right) + \tfrac{1}{2} k_2 (d - h - e_1)^2$$

$$\therefore \quad \frac{a(2h - e) + k_2(h - e)^2}{a_1(2d - 2h - e_1) + k_2(d - h - e_1)^2} = \frac{s_1}{s} \quad . \quad (117.),$$

from which equation the value of h may be found when the ratio $\frac{s_1}{s}$ is given.

If the elasticity of the material is perfect, then eq. (96.) determines the position of the neutral axis, where α is the area of the top flange; α_1, the area of the bottom flange, and so on, as in Art. 54.

If the vertical rib be neglected, then eq. (116.) becomes

$$\frac{ag}{a_1 g_1} = \frac{s_1}{s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (118.)$$

In cast-iron beams $s = 6\frac{1}{2}$ times s_1, and, moreover, a and a_1 are usually taken in the same ratio, hence in this case eq. (118.) becomes,

$$g = g_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (119.);$$

that is, the neutral axis nearly lies in the centre of the vertical depth of the beam. Without neglecting the vertical rib, eq. (117.) determines the position of the neutral line more accurately.

69. Mr. Hodgkinson and other writers on this subject have assumed in their calculations that cast iron is absolutely incompressible, and that the neutral axis lies at or near the upper edge of the beam. Now, independently of the foregoing investigation, the assumption of the incompressibility of cast-iron is inconsistent with the results of experiment.

ABSTRACT of Experiments on the Extension and Compression of Cast-iron Beams, given in the Report of the Government Commissioners for the Year 1849, pages 57. and 65.

Reduced Experiments. Bar 10 feet long, and 1 square inch in section of Blaenavon Iron, No. 2.

Weight laid on in lbs. (w).	Extension (e) in Inches.	Ratio $\frac{w}{e}$.
1,048	·0091	115,060
2,096	·0188	111,500
4,192	·0389	107,780
6,289	·0613	102,600
8,380	·0867	96,720
13,630	·1674	81,406
Weight laid on in lbs. (w).	**Compression (c) in Inches.**	**Ratio $\frac{w}{c}$.**
2,030	·01915	106,120
4,060	·03906	104,050
6,096	·05848	104,250
8,130	·07915	102,700
12,200	·11920	102,260
24,390	·24520	99,450
32,520	·34400	94,550
Crushing Force per Square Inch in Tons.	**Tensile Force per Square Inch in Tons.**	**Ratio of Crushing and Tensile Forces.**
7·406	49·1	1 : 6·577

The third column of this table shows, that with moderate forces the elongations and compression are pretty nearly in proportion to the forces producing these elongations and compressions, and moreover that, for *equal forces the compression is about equal to the elongation.*

Between the forces 1 and 3 tons, the formula $\frac{w}{e} = 107{,}000$ nearly expresses the ratio of the force of extension to the elongation which it produces; and $\frac{w}{e} = 89{,}000$ between 3 and 7

tons, or the ultimate strength; similarly $\frac{w}{c} = 103{,}000$ between 1 and 7 tons, and $\frac{w}{c} = 97{,}000$ between 7 tons and the ultimate strength. Hence for forces between 1 and 3 tons, we have for equal forces $e : c :: 103 : 107$; and between 3 and 7 tons, for equal weights, $e : c :: 103 : 89$.

For forces below $2\frac{1}{2}$ tons, the rate of compression is greater than that of extension; and for forces greater than $2\frac{1}{2}$ tons the rate of extension is greater than that of compression; but for forces near $2\frac{1}{2}$ tons the compressions are equal to the elongations. With these facts before us it seems strange that any philosophical mind should base calculations upon the hypothesis of the incompressibility of cast-iron.

Moment of Rupture, &c.

70. Here after the manner of Art. **17.**, we have,

$$\mathrm{I} = \tfrac{1}{3}\ \{b h^3 - (b - k_2)(h - e)^3\} \quad . \quad . \quad (120.),$$

$$\mathrm{I}_1 = \tfrac{1}{3}\ \{b_1 h_1^3 - (b_1 - k_2)(h_1 - e_1)^3\} \quad . \quad . \quad (121.);$$

$$\therefore \quad \mathrm{M} = s\mathrm{I} + s_1\mathrm{I}_1, \text{ or } \frac{\mathrm{S}\mathrm{I}}{h} + \frac{\mathrm{S}_1\mathrm{I}_1}{h_1}. \quad . \quad . \quad . \quad (122.)$$

If the elasticity is perfect, then $s = s_1$, and

$$\mathrm{M} = s\ (\mathrm{I} + \mathrm{I}_1) \text{ or } s_1\ (\mathrm{I} + \mathrm{I}_1)$$

$$= \frac{\mathrm{S}}{h}\ (\mathrm{I} + \mathrm{I}_1) \text{ or } \frac{\mathrm{S}_1}{h_1}\ (\mathrm{I} + \mathrm{I}_1) \quad . \quad . \quad . \quad (123.),$$

where h and h_1 are determined from eq. (96.) And M being determined by experiment, S may be found from this equation.

71. If the vertical rib be neglected, or $k_2=0$, then eqs. (120.) and (121.) become

$$I=\frac{b}{3}\left\{h^3-(h-e)^3\right\} \quad . \quad . \quad . \quad . \quad (124.);$$

$$I_1=\frac{b_1}{3}\left\{h_1{}^3-(h_1-e_1)^3\right\}. \quad . \quad . \quad . \quad (125.)$$

By substituting $g+\frac{e}{2}$ for h, and $g_1+\frac{e_1}{2}$ for h_1, we obtain precisely the same expressions for m and m_1 as those given in Art. **59.**: hence we also have, in this case, the relation of eq. (109.), the same quantities being neglected in the formula. Hence, also, the expression for the breaking weight given in eq. (110.) applies to beams having double flanges, with the same general formula of error.

Application of eq. (110.) *to cast-iron Beams.*

72. In order to calculate the limit of error, we shall first assume, in accordance with eq. (119.), that the neutral axis lies in the middle of the vertical line joining the centres of the top and bottom flanges. In Mr. Hodgkinson's 12th experiment, p. 428., of his edition of Tredgold, $e=\cdot 31$, $e_1=\cdot 66$, $d=5\cdot 125$,

$\therefore \quad G=5\cdot 125-\frac{\cdot 31+\cdot 66}{2}=4\cdot 64$, $g_1=2\cdot 32$, and $h_1=2\cdot 32+\cdot 33$ $=2\cdot 65$,

$$\therefore \quad \frac{g_1 G}{h_1 d}=\frac{2\cdot 32\times 4\cdot 64}{2\cdot 65\times 5\cdot 125}=\frac{11}{13} \text{ nearly,}$$

where there is an error of about $\frac{1}{5}$th.

Again taking the neutral axis at the lower edge of the upper flange, we have $d=5\cdot 125$, $G=5\cdot 125-\frac{\cdot 66+\cdot 31}{2}=4\cdot 64$, $g_1=$

$5\cdot 125-\cdot 31-\frac{\cdot 66}{2}=4\cdot 48$, $h_1=5\cdot 125-\cdot 31=4\cdot 815$,

$$\therefore \frac{g_1 \mathrm{G}}{h_1 d} = \frac{4{\cdot}31 \times 4{\cdot}64}{4{\cdot}815 \times 5{\cdot}125} = \frac{4}{5} \text{ nearly,}$$

where there is an error of about $\frac{1}{5}$th, as before.

These calculations show that the formula, Art. **59.**, theoretically considered, is not so applicable to beams of this kind, as to the tubular beams. However, although $\frac{g_1 \mathrm{G}}{h_1 d}$ may not be equal to unity, yet it will very nearly be a constant for all beams which differ but little from similarity. We shall now inquire how far these theoretical deductions agree with the results of experiment.

73. *Mr. Hodgkinson's experiments on cast-iron beams with double flanges, as given in his edition of Tredgold on the Strength of Material,* Table 11., page 434.

With the view of ascertaining how far the results of the preceding articles coincide with the deductions of experiment, the following table presents the proportion of the depths of the beams to their breaking weights. In experiments 10. and 11., the breaking weights are reduced to the same distance between the supports as in experiments 1, 2, 3, 4, and 5.

No. of Experiments.	Depth.	Area Section.	Breaking Weight.	Breaking Weight reduced to unity of Section.	Proportion of Depth to reduced breaking Weight.
1	4·1	6·54	13,543	2,070	1 : 505
2	5·2	6·94	15,129	2,180	1 : 420
3	6·0	7·08	15,129	2,130	1 : 355
4	6·93	7·67	22,185	2,890	1 : 417
5	6·98	7·40	19,049	2,570	1 : 368
10	10·25	7·83	36,800	4,700	1 : 459
11	10·25	9·10	41,400	4,550	1 : 444

Here it will be observed that where the beams approach to

an identity of section, we have, as might be expected, a near approach to the equality of ratio, as for example, in the case of experiments 10. and 11., and also in 3. and 5. And where the beams approach to similarity of section, or to such modifications as are explained in Art. **44.**, we have, in accordance with the law of similar beams demonstrated in Art. **39.**, a near approach to an equality of ratio, as for example in the case of experiments 2. and 11. But where the beams are decidedly dissimilar we find a marked deviation from the law, as for example in experiments 1. and 5., and also in 3. and 10.; in such cases the variation amounts to about $\frac{1}{5}$th as determined in Art. **72.**

New Formulæ relative to Cast-iron Beams.

74. For similar beams the true formula of strength is that given in Art. **39.**, and when the beams are not similar, formula (114.), Art. **65.**, determines the breaking weight with considerable precision, viz.,

$$\mathrm{W} = \frac{4\,\mathrm{S}\,a\,\mathrm{G}^2}{l(\mathrm{G}+e)} \text{ or } \frac{4\,\mathrm{S}_1 a_1 \mathrm{G}^2}{l(\mathrm{G}+e_1)}$$

Where the constants S and S_1 are determined from the equations

$$\mathrm{S} = \frac{\mathrm{W}\,l(\mathrm{G}+e)}{4\,a\,\mathrm{G}^2}$$

$$\text{and } \mathrm{S}_1 = \frac{\mathrm{W}\,l(\mathrm{G}+e_1)}{4\,a_1\mathrm{G}^2}.$$

To determine S_1, we have from experiment 12., given by Mr. Hodgkinson (see Art. **72.**),

$e = {\cdot}31$, $e_1 = {\cdot}66$, $l = 54$, $a = {\cdot}72$, $a_1 = 4{\cdot}4$, $d = 5{\cdot}125$, $\mathrm{G} = 4{\cdot}64$, and $\mathrm{W} = 26084$ lbs.

$$\therefore\quad \mathrm{S}_1 = \frac{26084 \times 54 \times 5{\cdot}3}{4 \times 4{\cdot}4 \times 4{\cdot}64^2} = 19630 \text{ lbs.} = 8{\cdot}763 \text{ tons.}$$

Taking the mean value of these constants from Mr. Hodgkinson's experiments, we find $S_1=8\cdot2$.

Here the values of S and S_1 are very nearly the respective resistances of cast-iron to compression and extension. The error in this formula is the value of $\frac{e^2}{12g^2}$ compared with unity. Hence, taking the dimensions of the beam just referred to, we have,

$$\frac{e^2}{12g^2}=\frac{\cdot31^2}{12\times2\cdot32^2}=\cdot0014=\frac{1}{710}\text{ nearly,}$$

or the error is $\frac{1}{710}$th part of the quantity retained. Hence the formula for the breaking weight is

$$W=\frac{32\cdot8\,a_1G^2}{l(G+e_1)}$$

where the value of $G=d-\frac{e+e_1}{2}=\frac{1}{2}(2d-e-e_1)$, making this substitution, and omitting the decimal part of the constant, we find

$$W=\frac{16a_1(2d-e-e_1)^2}{l(2d+e_1-e)}\text{ tons.}\quad\ldots\quad(126.)$$

75. Let us now apply this formula to some examples, with the view of comparing it with Mr. Hodgkinson's.

1. In experiment 11., Table II., page 434. of Mr. Hodgkinson's edition of Tredgold, $a_1=5\cdot7$, $d=10\frac{1}{4}$, $e=\cdot33$, $e_1=\cdot75$, $l=108$, then, by the proposed formula,

$$W=\frac{16\times5\cdot7\times19\cdot4^2}{108\times21\cdot92}=14\cdot5\text{ tons.}$$

Now the breaking weight of this beam is 14·38 tons.

By Mr. Hodgkinson's formula

$$W = \frac{26\, a_1 d}{l} = \frac{26 \times 5 \cdot 7 \times 10\frac{1}{4}}{108} = 14 \text{ tons.}$$

2. In experiment 10., Table II.,

$$a_1 = 4 \cdot 72,\ d = 10\tfrac{1}{4},\ e = \cdot 27,\ e_1 = \cdot 77,\ l = 108,$$

then by the proposed formula

$$W = \frac{16 \times 4 \cdot 72 \times 19 \cdot 46^2}{108 \times 21} = 12 \cdot 4 \text{ tons.}$$

By Mr. Hodgkinson's,

$$W = \frac{26 \times 10\frac{1}{4} \times 4 \cdot 72}{108} = 14 \cdot 2 \text{ tons.}$$

Now the breaking weight in this case is 12·8 tons.

3. Let us suppose the linear dimensions of a beam in all respects double that of experiment 12., then $e = 2 \times \cdot 31$, $e_1 = 2 \times \cdot 66$, $l = 2 \times 54$, $a_1 = 4 \times 4 \cdot 4$, and $d = 2 \times 5\frac{1}{8}$, then by Art. 32,

$$W = 2^2 \times 11 \text{ tons } 13 \text{ cwts.} = 46 \cdot 6 \text{ tons.}$$

Now by the proposed formula

$$W = \frac{16 \times 17 \cdot 6 \times 18 \cdot 56^2}{108 \times 21 \cdot 2} = 43 \text{ tons.}$$

And by Mr. Hodgkinson's formula

$$W = \frac{26 \times 17 \cdot 6 \times 10\frac{1}{4}}{108} = 43 \cdot 4 \text{ tons.}$$

Corollaries from Art. **70.**

76. If $s=6\frac{1}{2}\times s_1$, then eq. (122.) becomes

$$\text{M}=s_1\{6\tfrac{1}{2}\times\text{I}+\text{I}_1\}$$

$$=\frac{\text{S}_1}{h_1}\left\{6\tfrac{1}{2}\times\text{I}+\text{I}_1\right\}\quad\ldots\quad(127.),$$

where h and h_1 are found from eq. (117.)

If the vertical rib be neglected, or $k_2=0$, and $a_1=6\frac{1}{2}\times a$, $e_1=e$, $b_1=6\frac{1}{2}\times b$, then by eq. (119.), $g=g_1$, $\therefore$ $h_1=h=\frac{1}{2}d$, and by eqs. (124.) and (125.), $6\frac{1}{2}\times\text{I}=\text{I}_1$,

$$\therefore\quad \text{M}=\frac{2\,\text{S}_1\text{I}_1}{h_1}\text{ or }\frac{4\,\text{S}_1\text{I}_1}{d}\quad\ldots\ldots\quad(128.);$$

where in this case,

$$\text{I}_1=\frac{b_1}{3}\left\{h^3-(h-e)^3\right\}$$

77. If the material be considered incompressible, and the neutral axis to lie on the upper edge AB, then we in like manner find

$$\text{M}=\frac{\text{S}_1}{3d}\left\{b_1d^3-(b_1-k_2)(d-e_1)^3+(b-k_2)e^3\right\}.\ \ldots\ (129.)$$

Taking $b=k_2$ in this expression, we have

$$\text{M}=\frac{\text{S}_1}{3d}\left\{b_1d^3-(b_1-k_2)(d-e_1)^3\right\}..\ \ldots\ldots\ (130.)$$

The strongest Form of Cast-iron Beams.

78. It is obvious from eq. (126.) that W will increase with d, all the other dimensions remaining constant. Hence, it must be observed, that Mr. Hodgkinson's experimental beams only give the best form of the section when the depth is given.

Formulæ (69.) and (70.), Art. **42.**, enable us to compare the strengths of beams of different sections and lengths. For example, let us compare the strengths of Mr. Cubitt's beam, Art. **36.**, with that of Mr. Hodgkinson's, Art. **72.** Here by formula (70.) we shall reduce the breaking weight of the former beam to the same length and sectional area of the latter,

$$\therefore \quad W_1 = \left(\frac{6 \cdot 4}{18}\right)^{\frac{3}{2}} + \frac{16 \times 180}{54}$$

$$= 11 \cdot 32 \text{ tons,}$$

which is the breaking weight of a beam similar in section to Mr. Cubitt's, having the same sectional area and length as that of Mr. Hodgkinson's best beam. Now, the breaking weight of this beam is 11 tons 13 cwt. Hence, it appears, that the two beams are nearly of equal strength. If the top flange of Mr. Cubitt's beam had contained more material, so as to have made the top and bottom flanges in the ratio of 1 : $6\frac{1}{2}$, there could be no doubt but that this beam would have been stronger than the other one.

STRENGTH, &c., OF CYLINDRICAL BEAMS, &c.

CYLINDRICAL BEAMS.

79. Let NRBD be the transverse section of a cylindrical beam, BN the diameter, and O the centre of the circle. Let $r =$ the radius, $x = \mathrm{O}m$, $y = \mathrm{S}m$, then from the equation to the curve

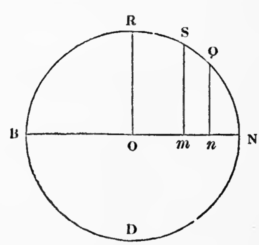


Fig. 12.

$$y = (r^2 - x^2)^{\frac{1}{2}},$$

$$\therefore \quad y^3 = (r^2 - x^2)^{\frac{3}{2}},$$

therefore substituting in eq. (34.), Art. **21.**

$$\text{I} = \tfrac{1}{3}\int_0^r (r^2 - x^2)^{\frac{3}{2}} dx$$

$$= \tfrac{1}{16}\pi r^4,$$

(see the Author's Calculus, Ex. 6. p. 183.) which is the momen of inertia of the quadrant NOR. Hence the moment of inertia of the whole circle about its diameter will be

$$\text{I} = \tfrac{1}{4}\pi r^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (131.);$$

but area section $= \pi r^2 = \text{K}$,

$$\therefore \quad \text{I} = \tfrac{1}{4}\text{K} r^2,$$

$$\therefore \quad \text{M} = \tfrac{1}{4} \cdot \frac{\text{S}}{r} \cdot \text{K} r^2$$

$$= \tfrac{1}{4}\text{SK}r \quad . \quad . \quad . \quad . \quad . \quad . \quad (132.),$$

which is the expression for the moment of rupture.

If the beam be supported at each end,

$$\text{W} = \frac{4\text{M}}{l}$$

$$= \frac{\text{SK}r}{l} \text{ or } \frac{\text{S}}{2} \cdot \frac{\text{K}d}{l} \quad . \quad . \quad . \quad . \quad (133.);$$

that is, *The breaking weight of a solid cylindrical beam is equal to the continued product of the sectional area, the depth, and a constant* $\left(\frac{\text{S}}{2}\right)$ *divided by the distance between the supports.*

80. If the beam be a hollow cylinder whose internal radius is r_1, then the moment of inertia will be equal to the dif-

ference of the moments of the external and internal circles, that is,

$$\text{I} = \tfrac{1}{4}\pi r^4 - \tfrac{1}{4}\pi r_1^{\,4}$$

$$= \frac{\pi}{4}(r^4 - r_1^{\,4}) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (134.);$$

putting K for the area of the section, we have

$$\text{I} = \frac{\pi}{4}(r^2 - r_1^{\,2})(r^2 + r_1^{\,2})$$

$$= \frac{\text{K}}{4}(r^2 + r_1^{\,2}) \text{ or } \frac{\text{K}}{16}(d^2 + d_1^{\,2}). \quad . \qquad (135.)$$

Hence we have

$$\text{M} = s\,\text{I} = \frac{s\text{K}}{16}(d^2 + d_1^{\,2}) \quad . \quad . \quad . \qquad (136.),$$

$$= \frac{\text{SK}}{8\,d}(d^2 + d_1^{\,2}). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (136^*.)$$

81. If the plate forming the hollow cylinder be thin as compared with the diameter, then $\frac{d_1}{d} = 1$ nearly

$$\therefore \quad \text{M} = \frac{\text{SK}d}{8}\left(1 + \frac{d_1^{\,2}}{d^2}\right)$$

$$= \frac{\text{SK}d}{4}, \text{ nearly.} \quad . \quad . \quad . \quad . \quad . \qquad (137.)$$

When the beam is loaded in the middle and supported at the extremities, we have

$$\frac{\text{W}l}{4} = \text{M},$$

$$\therefore \quad \text{W} = \frac{\text{SK}d}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (138.)$$

The mean value of the constant S, determined from the first six

experiments made by M. Fairbairn on cylindrical beams, is $14\frac{1}{2}$ tons. Hence we have the following rule for finding the strength of hollow cylindrical beams.

82. *In hollow cylindrical beams, formed of thin plates, the breaking weight, in tons, is equal to the continued product of the sectional area, the depth, and a constant* ($14\frac{1}{2}$) *divided by the distance between the supports.* These dimensions are all in inches.

The result is analogous to that derived for rectangular beams, Art. **59.**

CYLINDRICAL AND SQUARE BEAMS COMPARED.

83. *To determine the comparative strengths of a hollow cylindrical beam and a hollow square beam, having the same thickness of plates, and the diameter of the one equal to the side of the other.*

By eq. (82.), Art. **47.**, we have, for the square beam,

$$\text{M} = \frac{s\text{K}}{12}(d^2 + d_1^2).$$

And by eq. (136.), Art. **80.**, we have, for the cylindrical beam,

$$\text{M}_1 = \frac{s\text{K}_1}{16}(d^2 + d_1^2);$$

$$\therefore \quad \frac{\text{M}}{\text{M}_1} = \frac{4\text{K}}{3\text{K}_1},$$

but $\text{K} = d^2 - d_1^2$, and $\text{K}_1 = \frac{\pi}{4}(d^2 - d_1^2)$,

$$\therefore \quad \frac{\text{K}}{\text{K}_1} = \frac{4}{\pi},$$

$$\therefore \quad \frac{\text{M}}{\text{M}_1} = \frac{4}{3} \cdot \frac{4}{\pi} = \frac{16}{3\pi}.$$

Hence the comparative strengths of the two beams are as follows:

For equal thickness of plates, $\dfrac{\text{strength sq. beam}}{\text{strength cir. beam}} = \dfrac{16}{3\pi}$.

For equal areas, $\dfrac{\text{strength sq. beam}}{\text{strength cir. beam}} = \dfrac{16}{3\pi} \div \dfrac{4}{\pi} = \dfrac{4}{3}$.

With the same material, the square beam has $1\frac{1}{3}$ times the strength of the cylindrical one.

COMPARATIVE STRENGTHS OF CIRCULAR AND SQUARE CELLS.

84. Let KEGF be a circular cell, and ABCD a square one, where the areas of the sections are the same, the diameter

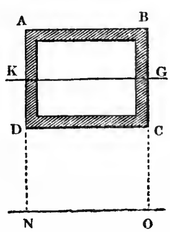


Fig. 13.

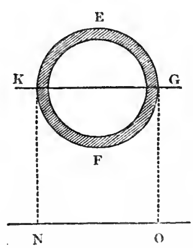


Fig. 14.

KG = the side AB = d, the lines KG passing through the centres of the two cells are at the same distance ($\bar{x}$) from the axis of moments, NO.

By Art **49.**,

Moment of rupture of the square cell, M,

$$= \frac{s\text{K}}{12}(d^2 + d_1{}^2 + 12\bar{x}^2).$$

By Arts. **80.** and **23.**, eqs. (136.) and (39.),

Moment of rupture of the circular cell, M_1,

$$= \frac{s\,K}{16}(d^2 + d_2^2 + 16\bar{x}^2),$$

where $d_2^2 =$ the square of the interior diameter $= d^2 - \frac{4}{\pi}K$.

Now M will be greater or less than M_1 according as

$$\frac{1}{12}(d^2 + d_1^2) \gtrless \frac{1}{16}(d^2 + d_2^2),$$

$$4d^2 + 4d_1^2 \gtrless 3d^2 + 3d_2^2,$$

$$d^2 + 4d_1^2 \gtrless 3\left(d^2 - \frac{4}{\pi}K\right),$$

$$d_1^2 + \frac{6}{\pi}K \gtrless (d^2 - d_1^2),$$

$$d_1^2 + \frac{6}{\pi}K \gtrless K,$$

but $\frac{6}{\pi}K$ is greater than K, therefore M is greater than M_1; that is, the square cell is stronger than the circular one.

OBSERVATIONS RELATIVE TO THE BEST FORM OF THE CELLS IN A TUBULAR BEAM.

85. It appears from the foregoing investigation, that with a given section of material the square form of cells is stronger than the circular form. Mr. H., however, found from experiment that the circular cells are stronger than the square ones when subjected to a simple crushing force exerted equally

over their sections; and he therefore infers that the former are better adapted to a tubular beam than the latter. This *inference* admits of dispute, for it is contrary to theory.

The cells in a transverse strain undergo a different kind of strain to what they are subjected to in a *simple crushing* force equally distributed over the section of the tube. In this case all the parts of the section are equally compressed, and it is reasonable to conclude that the best form of the cell will be that in which the material is equally distant from the axis of pressure; but the case of transverse strain is very different; the upper edge undergoes the greatest strain, and of the other parts, that which is nearest the neutral axis of the beam undergoes the least: in this case, therefore, the material in the square cells is symmetrically distributed with respect to the axis of pressures—the neutral axis of the beam. Be this as it may, we are not disposed to give up a principle established by theory until some *direct experiments* should prove the contrary. In a rectangular cell the weakest part is obviously at the corners or points of junction of the plates; but if these points are strengthened by angle irons, as they are in the Conway Tube, the cells are not only rendered more perfect, but a new important auxiliary element is introduced into their structure.

WHY THE CELLULAR STRUCTURE EXHIBITS SUCH STRENGTH.

86. Let AC and EG represent the sections of two beams undergoing transverse strain, *in all respects the same*, excepting that in the former case the material at AB is composed of horizontal plates in contact with each other, whereas in the latter case the material is arranged in the form of cells; the horizontal plates being connected by vertical plates or ribs, *a*, *b*, *c*, *d*, &c.

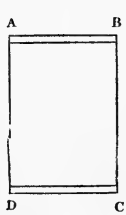


Fig. 15.

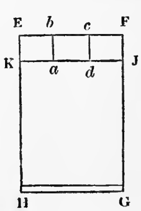


Fig. 16.

When thin plates of wrought iron are subject to compression, they double or crumple up long before the material would be destroyed under ordinary circumstances by crushing. In the first case of construction (*fig.* 15.), no means are employed to counteract this tendency to double; but in the latter case it is different. Here the horizontal plates, as well as the vertical ones, have a tendency to double up; but the direction of this tendency in the former is in a plane at right angles to the plane in which the latter would take place; that is to say, the horizontal plates EF and KJ tend to crumple vertically, while the vertical plates *ab*, *cd*, &c. tend to crumple horizontally. Again, the direction of greatest *strength* in the vertical plates is in the vertical line, and at the same time the direction of *weakness* of the horizontal plates is also in the vertical line, and *vice versa:* hence the horizontal plates EF and KJ are prevented from crumpling by the vertical plates *ab*, *dc*, &c., and *vice versa.* It is evident, that the horizontal plates EF and KJ could not crumple without exerting a vertical strain upon the vertical plates *ab*, *dc*, &c.: and, in like manner, the vertical plates could not crumple without exerting a horizontal strain upon the horizontal plates.

It appears, therefore, that the object of the cellular structure is to counteract the tendency which thin plates, acted upon by a compressive force, have to crumple, and thus to cause the tubular beam to be subjected to the same laws of transverse strain as an ordinary beam. Hence it is not necessary that the bottom part of the tube should have a cellular structure.

MOMENT OF INERTIA OF CIRCULAR SPACES, ETC.

87. *To determine the moment of inertia of a semicircle about an axis passing through its centre of gravity parallel to the diameter.*

Let ABR be a semicircle, KG the axis passing through its centre of gravity parallel to the diameter AB. (See *fig.* 17.)

Put $\bar{x}=$ the distance of KG from AB.

By eq. (131.) the moment of inertia of a semicircle about its diameter is $\frac{1}{8}\pi r^4$; therefore, by eq. (40.), Art. **23.**, the moment about the axis through the centre of gravity is

$$\mathrm{I}_o = \frac{1}{8}\pi r^4 - \text{area} \times \bar{x}^2$$

$$= \frac{1}{8}\pi r^4 - \frac{1}{2}\pi r^2 \bar{x}^2 \quad . \quad . \quad . \quad . \quad (139.);$$

but $x = \frac{4r}{3\pi}$, (see the Author's Calculus, p. 226.),

$$\therefore \quad \mathrm{I}_o = r^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right)$$

$$= \cdot 11 r^4, \text{ nearly.} \quad . \quad . \quad . \quad (140.)$$

88. *To determine the moment of inertia of a semicircle about any axis* NO *parallel to the diameter* AB.

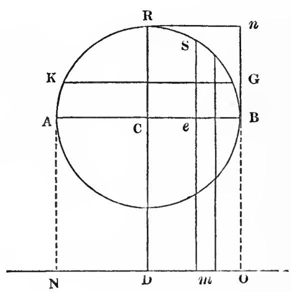


Fig. 17.

Let $e=$ NA $=$ OB, $\mathrm{I}_o=$ the moment of inertia of the semicircle about the axis KG, and $\mathrm{I}_x=$ the moment about NO; then I_o is given in eq. (139.); therefore, by eq. (39.) Art. **23.**,

$$\mathrm{I}_x = \mathrm{I}_o + \text{area ARB} \times \mathrm{OG}^2$$

$$= \frac{1}{8}\pi r^4 - \frac{1}{2}\pi r^2 \bar{x}^2 + \frac{1}{2}\pi r^2(\bar{x}+e)^2$$

$$= \frac{1}{8}\pi r^4 + \frac{1}{2}\pi e^2 r^2 + \pi e r^2 \bar{x}$$

$$= \frac{\pi r^2}{8}(r^2 + 4e^2) + \frac{4er^3}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141.);$$

which is the moment of inertia of a semicircle about any axis parallel to its diameter.

Or thus by a direct investigation:

Let Se be a small element of surface, C the centre of the circle; put $x = \text{C}e = \text{D}m$, $y = \text{S}e$, $\Delta x =$ the breadth of the elementary rectangle Se, then, by Art. **17.**, page 13.,

$$\text{The moment of inertia of the rectangle } \text{S}e = \frac{\Delta x}{3}\left\{(y+e)^3 - e^3\right\}$$

$$= \frac{1}{3}\Delta x(y^3 + 3ey^2 + 3e^2y).$$

Now, by the equation to the circle $y^2 = r^2 - x^2$, therefore, moment inertia of the quadrant BCR

$$= \frac{1}{3}\int_0^r\left\{(r^2 - x^2)^{\frac{3}{2}} + 3e\,(r^2 - x^2) + 3e^2\,(r^2 - x^2)^{\frac{1}{2}}\right\}dx$$

$$= \frac{\pi r^4}{16} + \frac{2er^3}{3} + \frac{e^2\pi r^2}{4}$$

$$= \frac{\pi r^2}{16}(r^2 + 4e^2) + \frac{2er^3}{3}$$

$\therefore$ moment inertia semicircle ABR $= \dfrac{\pi r^2}{8}(r^2 + 4e^2) + \dfrac{4er^3}{3}$,

as before determined.

Now the moment inertia of the whole circle about NO = moment about its diameter + area $\times$ DC^2

$$= \frac{1}{4}\pi r^4 + \pi r^2 e^2 = \frac{\pi r^2}{4}(r^2 + 4e^2) \quad . \quad . \quad (142.)$$

$\therefore$ moment semicircle lying below the diameter AB = moment whole circle — moment semicircle ABR

$$= \frac{\pi r^2}{8}(r^2 + 4e^2) - \frac{4er^3}{3}. \quad . \quad . \quad . \quad (143.)$$

If in this case $e = \text{DR} = e + r$; then we have moment inertia semicircle ABR placed in an inverted position

$$= \frac{\pi r^2}{8}\left\{r^2 + 4\,(e+r)^2\right\} - \frac{4\,r^3\,(e+r)}{3} \quad . \; . \qquad (144.)$$

89. *The moment of inertia of a circle about an axis forming a tangent to its circumference is five times the moment when the axis passes through the centre.*

In eq. (142.) let $e = r$, then

$$\text{moment inertia about the tangent} = \frac{5\,\pi\,r^4}{4}$$

$= 5 \times$ moment inertia about the diameter, by eq. (131.)

90. *The difference of the moments of inertia of the semicircle* ABR *and* ABV *about* NO *as an axis, is equal to the moment of inertia of the rectangle* ABON *about* AN *as an axis.* (See *fig.* 17.)

Subtracting eq. (143.) from eq. (141.), we obtain the difference $\frac{8\,e\,r^3}{3} = \frac{1}{3}e\,(2\,r)^3 =$ moment inertia rectangle ABON about the axis AN.

91. *To determine the moment of inertia of the space* BSRn *about the axis* NO. See *fig.* 17.

Let q be put for the moment of inertia of the space BSRn, then

Moment inertia BSRn = moment square RnBC — moment inertia quadrant BCR,

$$\therefore \quad q = \frac{r}{3}\left\{(e+r)^3 - e^3\right\} - \frac{\pi\,r^2}{16}\,(r^2 + 4\,e^2) - \frac{2\,e\,r^3}{3} \quad . \; . \qquad (145.)$$

92. *To determine the moment of inertia of the section* ABDC *of a beam with double flanges, having the quadrants* (rs) *of circles forming the interior angles.*

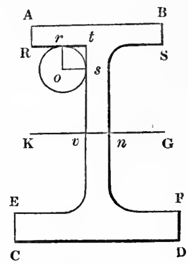


Fig. 18.

Let b, b_1 = AB and CD respectively;

h, h_1 = the distances of AB and CD respectively from the neutral axis KG;

h' = KR the distance of RS from KG;

k_2 = the thickness of the vertical rib;

$r = os = or$, the radius of the circles forming the quadrants rs, &c.;

$e = vs$, the distance of the centre of the circle from KG;

Q = the moment of inertia of the rectangular parts above KG;

q = the moment of inertia of the circular portion rts above KG; and so on to these symbols accented for the parts below KG.

By eq. (120.), Art. **70.**,

$$Q = \frac{1}{3}\left\{bh^3 - (b - k_2)h'^3\right\}. \quad . \quad . \quad (146.)$$

And by eq. (145.), Art. **91.**, q. is given,

$$\therefore \quad I = Q + 2q, \text{ and } I_1 = Q_1 + 2q_1;$$

if the material is perfectly elastic, we have

$$M = \frac{S}{h}(Q + 2q + Q_1 + 2q_1) \quad . \quad . \quad (147.);$$

and when the beam is supported at each extremity,

$$\text{W} = \frac{4\,\text{S}}{l h}\left\{\text{Q} + 2q + \text{Q}_1 + 2q_1\right\} \quad . \quad . \quad (148.);$$

where it is to be observed that Q is given in eq. (146.), q in eq. (145.), and so on to Q_1 and q_1.

The neutral axis is determined from eq. (13.).

STRENGTH, &c. OF ELLIPTICAL BEAMS, &c.

93. Let NRBD (see *fig.* 3.) represent the transverse section of an elliptical beam, ON the semi-minor diameter, and OR the semi-major diameter. Let $b = \text{ON} = \text{OB}$, $a = \text{OR}$, $x = \text{O}m$, $y = \text{S}m$; then, by the equation to the curve,

$$y = \frac{a}{b}(b^2 - x^2)^{\frac{1}{2}};$$

therefore by substitution in eq. (34.), Art. **21.**,

$$\text{I} = \frac{1}{3} \cdot \frac{a^3}{b^3}\int_0^b (b^2 - x^2)^{\frac{3}{2}}\, dx$$

$$= \frac{1}{16}\pi b a^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (149.),$$

which is the expression for the quadrant NOR of the ellipse. Hence the moment of inertia of the whole ellipse about its minor diameter will be

$$\text{I} = \frac{1}{4}\pi b a^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (150.),$$

but area section $=\pi a b=\text{K}$,

$$\therefore \quad \text{I}=\frac{1}{4}\text{K}a^2$$

$$\therefore \quad \text{M}=\frac{\text{S}}{4a}\,.\,\text{K}a^2=\frac{1}{4}\text{SK}a$$

$$=\frac{1}{8}\text{SK}d. \quad . \quad . \quad . \quad . \quad . \quad . \quad (151.)$$

where d is put for the whole depth of the beam or the major axis of the ellipse.

94. If the beam be supported at each end,

$$\text{W}=\frac{\text{SK}a}{l} \text{ or } \frac{\text{S}}{2}\,.\,\frac{\text{K}d}{l} \quad . \quad . \quad . \quad (152.);$$

that is, *the breaking weight of a solid elliptical beam is equal to the continued product of the sectional area, the depth, and a constant* $(\frac{\text{S}}{2})$ *divided by the distance betweeu the supports.*

It will be observed that this formula applies to all elliptical beams, whatever may be the ratio of the depth to the breadth.

Hollow elliptical Beams.

9. Let a_1 and b_1 be put for the semi-axes of the interior ellipse; then from eq. (150.),

$$\text{I}=\frac{\pi}{4}(b a^3-b_1 a_1{}^3),$$

$$\therefore \quad \text{M}=\frac{\text{S}\pi}{4a}(b a^3-b_1 a_1{}^3) \quad . \quad . \quad (153.);$$

taking $\frac{a}{b} = \frac{a_1}{b_1}$, we find,

$$M = \frac{S\pi}{4b}(b^2a^2 - b_1^2a_1^2)$$

$$= \frac{S\pi}{4b}(ba - b_1a_1)(ba + b_1a_1)$$

$$= \frac{SK}{4b}(ba + b_1a_1)$$

$$= \frac{SKa}{4}(1 + \frac{b_1a_1}{ba})$$

$$= \frac{SKa}{2} \text{ or } \frac{SKd}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (154.)$$

by taking $\frac{b_1a_1}{ba} = 1$, which it is very nearly when the plate forming the hollow beam is thin as compared with either diameter. In experiment 19^n, made by Mr. Fairbairn on elliptical tubes, $\frac{b_1a_1}{ba} = \cdot 98$.

96. When the beam is loaded in the middle and supported at the extremities, we have

$$W = \frac{SKd}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (155.);$$

the mean of the constant S, determined from Mr. Fairbairn's experiments on elliptical tubes, is 15 tons. Hence we have the following rule for finding the strength of hollow elliptical beams.

In hollow elliptical beams formed of thin plates, the breaking weight in tons is equal to the continued product of the sectional

area, the depth, and a constant (15.), *divided by the distance between the supports.* These dimensions are all in inches.

This result is analogous to that derived for rectangular and cylindrical beams, Arts. **39.** and **82.**

97. *To determine the strength, &c., of a beam having a rectangular section,* ABDC (see *fig.* 18.), *hollowed at the sides by the semi-ellipses,* RvE *and* SnF.

Let $m, n =$ the major and minor diameters of the ellipse respectively.

Here, by eq. (78.), the moment of inertia of the solid rectangle, ABDC, about the axis, KG, passing through the middle of the beam, is $\frac{1}{12} b d^3$, where $b = \text{AB} = \text{CD}$ the breadth of the beam, and $d = \text{AC} = \text{BD}$, the depth. And the moment of inertia of the ellipse is given in eq. (150.):

$$\therefore \quad \text{I} = \frac{1}{12} b d^3 - \frac{1}{64} \pi n m^3$$

$$\therefore \quad \text{M} = \text{S} \left\{ \frac{1}{6} b d^2 - \frac{1}{32} \frac{\pi n m^3}{d} \right\} \quad . \quad . \quad (156.),$$

which expresses the moment of rupture of the beam.

If $m = d$, then

$$\text{M} = \text{S} d^2 \left\{ \frac{1}{6} b - \frac{1}{32} \pi n \right\}. \quad . \quad . \quad (157.)$$

If $n = b$, then

$$\text{M} = \frac{\text{S}(16 - 3\pi)}{96} \,.\, b d^2. \quad . \quad . \quad . \quad (158.)$$

MOMENTS OF INERTIA, &c. OF TRIANGULAR AND TRAPEZOIDAL SURFACE.

98. *To determine the moment of inertia of a triangle* ABC *about its base* AB.

In the right-angled triangle ADC, let $c =$ AD, $a =$ CD, $x =$ AM, $y =$ MP, then from the similar triangles ADC and AMP, we have

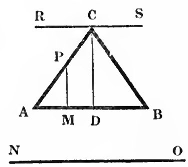


Fig. 19.

$$\text{AD} : \text{DC} :: \text{AM} : \text{MP}$$

$$c : a :: x : y, \therefore y = \frac{ax}{c},$$

substituting in eq. (34.), we have for the moment of inertia of the triangle ADC,

$$\text{I} = \frac{1}{3} \int_0^c y^3 dx = \frac{a^3}{3c^3} \int_0^c x^3 dx.$$

$$= \frac{1}{12} c a^3.$$

Putting $c_1 =$ DB, and $b =$ AB, we have, in like manner, the moment of inertia of the triangle DBC $= \frac{1}{12} c_1 a^3$,

$$\therefore \text{I} = \frac{1}{12} c a^3 + \frac{1}{12} c_1 a^3$$

$$= \frac{1}{12} b a^3 \quad \ldots \ldots \ldots \quad (159.),$$

which expresses the moment of inertia of the triangle ABC about AB as an axis.

99. *To find the moment of inertia of a triangle about an axis passing through its centre of gravity and parallel to the base.*

Here the distance of the centre of gravity from the base is $\frac{a}{3}$, hence we have by eq. (40.)

$$I_o = \frac{1}{12} b a^3 - \frac{1}{2} a b \times \left(\frac{a}{3}\right)^2$$

$$= \frac{1}{36} b a^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (160.)$$

100. *To find the moment of inertia of a triangle about an axis* RS *passing through the vertex* C *parallel to the base* AB. (See *fig.* 19.)

Here the distance of the axis RS from that passing through the centre of gravity is $\frac{2a}{3}$, hence we have by eq. (39.)

$$I = I_0 + \frac{1}{2} a b \times \left(\frac{2a}{3}\right)^2$$

$$= \frac{1}{36} b a^3 + \frac{2}{9} b a^3$$

$$= \frac{1}{4} b a^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (161.)$$

101. *To find the moment of inertia of a triangle about any axis* NO *parallel to the base* AB. (See *fig.* 19.)

Let a_1 = the distance of NO from AB,

I_x = the moment of inertia of the triangle ABC about NO.

In this case, the distance of N O from the axis passing through the centre of gravity, is $a_1 + \frac{a}{3}$,

$$\therefore \quad I_x = I_0 + \frac{1}{2} a b \left(a_1 + \frac{a}{3}\right)^2$$

$$= \frac{1}{36} b a^3 + \frac{1}{18} a b (3 a_1 + a)^2$$

$$= \frac{a b}{36}\left\{a^2 + 2(3 a_1 + a)^2\right\}. \quad . \quad (162.)$$

When $a_1 = -a$ this expression becomes the same as (161.)

102. *To find the moment of inertia of a triangle when the vertex* C *lies towards the axis* N O.

Let $a_1 = $ A N $=$ B O the distance of the base from N O,

$b = $ A B, the base of the triangle,

$a = $ the perpendicular height of the triangle,

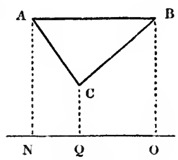


Fig. 20.

Here the distance of the centre of gravity of the triangle from N O is $a_1 - \frac{a}{3}$,

$$\therefore \quad I_x = I_0 + \frac{1}{2} a b \left(a_1 - \frac{a}{3}\right)^2$$

$$= \frac{1}{36} b a^3 + \frac{1}{18} a b (3 a_1 - a)^2$$

$$= \frac{a b}{36}\left\{a^2 + 2(3 a_1 - a)^2\right\}. \quad . \quad . \quad (163.)$$

102. *To find the moment of inertia of a trapezoid* ABDC *about its base* AB *as an axis.*

Draw CE *parallel to* DB, *and let* $b =$ AB, $b_1 =$ CD, *and* a = *perpendicular distance between* AB *and* CD.

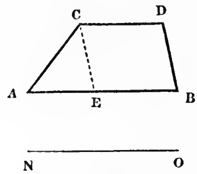


Fig. 21.

$$\text{I} = \text{moment inertia triangle AEC} + \text{moment inertia parallelogram CDBE}$$

$$= \frac{1}{12}(b - b_1)a^3 + \frac{1}{3}b_1 a^3$$

$$= \frac{1}{12}a^3(b + 3b_1) \; . \; . \; . \; . \; . \; . \; . \; . \quad (164.)$$

103. *To find the moment of inertia of the trapezoid* ABDC *about any axis* NO *parallel to the side* AB *or* CD.

Put $a_1 =$ the distance between AB and NO.

Here the moment of inertia of the triangle AEC is given in eq. (162.), and that of the parallelogram EBDC in Art. **17.**; proceeding, therefore, as in the last case, we have

$$\text{I}_x = \frac{1}{36}a(b - b_1)\left\{a^2 + 2(3a_1 + a)^2\right\} + \frac{1}{3}b_1\left\{(a + a_1)^3 - a^3\right\}. \quad (165.)$$

104. *To determine the strength of a square solid beam* ABCD *having its diagonal* AB *placed horizontally.*

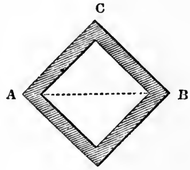


Fig. 22.

Let $c =$ AB, the diagonal,

$d =$ AD, the side,

By eq. (159.)

$$\text{Moment inertia triangle ABC} = \frac{1}{12} c \left(\frac{c}{2}\right)^3$$

$$= \frac{1}{96} c^4,$$

$$\therefore \quad \text{Moment inertia square ACBD} = \frac{1}{48} c^4 \quad . \quad (166.)$$

$$\therefore \quad \text{M} = \frac{\text{S}}{\frac{1}{2}c} \cdot \frac{1}{48} \cdot c^4 = \frac{\text{S}c^3}{24}$$

$$= \frac{\text{SK}c}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (167.);$$

that is, *the strength varies as the product of the sectional area and the depth.*

106. Now $c^2 = 2d^2$, therefore by substitution, eqs. (166.) and (167.) become

$$\text{I} = \tfrac{1}{12} d^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (168.),$$

$$\text{and M} = \frac{\sqrt{2}}{12} \text{SK}d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (169.)$$

Here the expression of eq. (168.) is the same as it would be if the side of the beam were horizontal; moreover, when the side AD is horizontal, $\text{M} = \frac{1}{6}\text{SK}d$; therefore the strength in the former case is to that of the latter as $\sqrt{2} : 2$.

107. *To determine the strength of a square hollow beam* ABCD *having its diagonal,* AB, *placed horizontally.* (See *fig.* 22.)

I_o = moment inertia exterior square — moment inertia interior square.

$$= \tfrac{1}{12} d^4 - \tfrac{1}{12} d_1^4.$$

$$= \tfrac{1}{12}(d^4 - d_1^4).$$

$$= \frac{\text{K}}{12}(d^2 + d_1^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (170.)$$

which is the same expression as that given in eq. (82.) for the cell with its side placed horizontally.

$$\text{M} = \frac{\text{S}}{\frac{1}{2}c} \cdot \frac{\text{K}}{12}(d^2 + d_1^2)$$

$$= \frac{\sqrt{2}}{12d} \cdot \text{SK}(d^2 + d_1^2)$$

Comparing this with eq. (84.) we find, that the strengths of the beam in the two positions to be as $\sqrt{2} : 2$.

STRENGTH, &c. OF PARABOLIC BEAMS.

108. Let RNDB be the section of a beam (see *fig.* 3.) where BRN and BDN are two equal parabolas, having their vertices at R and D.

Let $e = \text{ON} = \text{OB}$, $c = \text{OR} = \text{OD}$, $d = \text{RD}$ the whole depth of the beam, $b = \text{BN}$ the whole breadth, $x = \text{O}m$, $y = \text{S}m$.

By the equation to the curve,

$$x^2 = p\,(c - y),$$

$$\therefore \quad y^3 = \frac{1}{p^3}\,(pc - x^2)^3,$$

$$= \frac{c^3}{e^6}\,(e^2 - x^2)^3,$$

therefore by substitution in eq. (34.), we have for the moment of inertia of the space NOR,

$$\text{I} = \frac{1}{3}\int_0^e y^3\,dx,$$

$$\text{I} = \frac{c^3}{3\,e^6}\int_0^e (e^2 - x^2)^3\,d\,x$$

$$= \frac{16}{105}e\,c^3,$$

therefore the moment of inertia of the whole section is

$$\text{I} = \frac{16 \times 4}{105}e\,c^3$$

$$= \frac{4}{105}b\,d^3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (171.)$$

$$\text{and M} = \frac{4}{105}\cdot\frac{\text{S}}{\frac{1}{2}d}\cdot b\,d^3$$

$$= \frac{8}{105}\text{S}\,b\,d^2$$

$$= \frac{12}{105}\text{S}\,\text{K}\,d \quad . \quad . \quad . \quad . \quad . \quad . \qquad (172.);$$

that is, *the strength varies as the product of the sectional area and the depth.* Hence it appears that this remarkable law not only holds mathematically true for solid rectangular beams, but also for all beams whose transverse section has the form of a conic section.

109. Comparing eq. (172.) with eq. (28.), we find that the strength of a parabolic beam is to that of a rectangular one, the breadth and depth being the same in both beams, as 72 is to 105, or as 1 is to $1\frac{1}{2}$ nearly.

110. *To determine the strength, &c. of a beam having a rectangular section,* ABDC (see *fig.* 15.), *hollowed at the sides by parabolas,* RvE *and* SnF.

Let b_1 = the sum of the bases of the semi-parabolas Rrsv and Sn;

d_1 = RE = SF, the sum of their perpendicular heights;

b = AB = CD, the breadth of the beam;

d = AC = BD, the depth of the beam.

The moment of inertia of the solid rectangle ABDC, about the axis KG, passing through the middle of the beam is $\frac{1}{12}bd^3$. And the moment of inertia of the parabolic spaces is given in eq. (171.)

$$\therefore \quad \mathrm{I} = \frac{1}{12}bd^3 - \frac{4}{105}b_1 d_1^3. \quad . \quad . \quad (173.)$$

If $d_1 = d$, and $b_1 = b$, then

$$\mathrm{M} = \frac{\mathrm{S}}{\frac{1}{2}d}\left(\frac{1}{12}bd^3 - \frac{4}{105}bd^3\right)$$

$$= \frac{19}{210}\mathrm{S}bd^2;$$

but the section of the material, $\mathrm{K} = \frac{1}{3}bd$,

$$\therefore \quad \mathrm{M} = \frac{19}{70}\mathrm{SK}d \quad . \quad . \quad . \quad . \quad (174.);$$

that is, *the strength varies as the product of the sectional area and the depth.*

This principle is not restricted to parabolic sections, for in eq. (158.) we have

$$\mathrm{K} = bd\left(1 - \frac{\pi}{4}\right);$$

therefore, by substitution in eq. (158.), we find

$$\mathrm{M} = \frac{16 - 3\pi}{24\,(4 - \pi)}\mathrm{SK}d. \quad . \quad . \quad . \quad (175.),$$

where the same law is observed.

DOCTRINE OF SIMILAR BEAMS APPLIED TO PARTICULAR FORMS.

111. In the beam represented by *fig.* 11., we have, by eq. (96.), for the position of the neutral axis

$$\bar{x} = \frac{\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1}{\mathrm{K}},$$

where α and α_1 are put for the areas of the top and bottom flanges respectively, and so on as in Art. **54.**

$$\therefore \bar{x}' = \frac{\alpha'\alpha_1' + \beta'\beta_1' + \gamma'\gamma_1'}{\mathrm{K}'};$$

but $\alpha' = r^2\alpha$, &c., $\alpha_1' = r^2\alpha_1$, &c., $\mathrm{K}' = r^2\mathrm{K}$,

$$\therefore \bar{x}' = \frac{r^3(\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1)}{r^2\mathrm{K}}$$

$$= r\bar{x};$$

that is, the neutral axes divide the vertical depths of the beams proportionably.

Again, by eq. (120.),

$$\mathrm{I} = \frac{1}{3}\left\{bh^3 - (b - k_2)(h - e)^3\right\},$$

$$\therefore \mathrm{I}' = \frac{1}{3}\left\{b'h^{3\prime} - (b' - k_2')(h' - e')^3\right\};$$

but $b' = rb$, $k_2' = rk_2$, $e' = re$, and by the foregoing result $h' = rh$,

$$\therefore \mathrm{I}' = \frac{r^4}{3}\left\{bh^3 - (b - k_2)(h - e)^3\right\}$$

$$= r^4\mathrm{I}.$$

Similarly, $\mathrm{I}_1' = r^4\mathrm{I}_1$.

Hence, by eq. (123.), we have

$$\text{M} = \frac{\text{S}}{h}(\text{I} + \text{I}_1),$$

$$\therefore \text{M}' = \frac{\text{S}}{h'}(\text{I}' + \text{I}_1')$$

$$= \frac{\text{S}}{rh}(r^4\text{I} + r^4\text{I}_1)$$

$$= r^3\text{M},$$

which coincides with the more general formula given in Art. **31.**

By proceeding as in Art. **32.**, the relation $\text{W}' = r^2\text{W}$ is established.

112. Taking the hollow cylindrical beams, Art. **80.**, we have from eq. (136.),

$$\text{M} = \frac{\text{SK}}{8d}(d^2 + d_1^2),$$

$$\text{M}' = \frac{\text{SK}'}{8d'}(d'^2 + d'^2_1);$$

but $\text{K}' = r^2\text{K}$, $d' = rd$, $d_1' = rd_1$,

$$\therefore \text{M}' = \frac{\text{S}r^2\text{K}}{8rd}(r^2d^2 + r^2d_1^2)$$

$$= r^3\text{M}.$$

And so on as before for the relation, $\text{W}' = r^2\text{W}$.

113. Taking the beam represented by *fig.* 18., Art. **92.**, we have, by eq. (148.),

$$\text{W} = \frac{4\text{S}}{lh}(\text{Q} + 2q + \text{Q}_1 + 2q_1),$$

$$\therefore \text{W}' = \frac{4\text{S}}{l'h'}(\text{Q}' + 2q' + \text{Q}_1' + 2q_1').$$

From what has been proved in Art. **111.**, we have

$$Q' = r_1^4 Q,$$

where r_1 in this case is employed to express the linear ratio of the dimensions of the beams.

From eq. (145.),

$$q = \frac{r}{3}\left\{(e+r)^3 - e^3\right\} -, \text{ \&c.},$$

$$\therefore\ q' = \frac{r'}{3}\left\{(e'+r')^3 - e'^3\right\} -, \text{ \&c.},$$

but $r' = r_1 r$, $e' = r_1 e$, &c.,

$$\therefore\ q' = r_1^4 q,$$

and so on to the values of Q_1 and q_1,

$$\therefore\ W' = \frac{4S}{r_1^2 lh}\left\{r_1^4 Q + 2r_1^4 q + r_1^4 Q_1 + 2r_1^4 q_1\right\}$$

$$= r^2 W.$$

In like manner any other particular case may be established.

THE END.

LONDON:
SPOTTISWOODES and SHAW,
New-street-Square.